

# AN INDEFINITE CONCAVE-CONVEX EQUATION UNDER A NEUMANN BOUNDARY CONDITION I

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ABSTRACT. We investigate the problem

$$(P_\lambda) \quad -\Delta u = \lambda b(x)|u|^{q-2}u + a(x)|u|^{p-2}u \text{ in } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,$$

where  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^N$  ( $N \geq 2$ ),  $1 < q < 2 < p$ ,  $\lambda \in \mathbb{R}$ , and  $a, b \in C^\alpha(\overline{\Omega})$  with  $0 < \alpha < 1$ . Under some indefinite type conditions on  $a$  and  $b$  we prove the existence of two nontrivial non-negative solutions for  $|\lambda|$  small. We characterize then the asymptotic profiles of these solutions as  $\lambda \rightarrow 0$ , which implies in some cases the positivity and ordering of these solutions. In addition, this asymptotic analysis suggests the existence of a loop type subcontinuum in the non-negative solutions set. We prove in some cases the existence of such subcontinuum via a bifurcation and topological analysis of a regularized version of  $(P_\lambda)$ .

## 1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$  ( $N \geq 2$ ) with smooth boundary  $\partial\Omega$ . This article is concerned with existence, non-existence, and multiplicity of non-negative solutions for the problem

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda b(x)|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$  is the usual Laplacian in  $\mathbb{R}^N$ ,
- $\lambda \in \mathbb{R}$ ,
- $1 < q < 2 < p < \infty$ ,
- $a, b \in C^\alpha(\overline{\Omega})$  with  $\alpha \in (0, 1)$ ,
- $\mathbf{n}$  is the unit outer normal to the boundary  $\partial\Omega$ .

By a solution of  $(P_\lambda)$  we mean a classical solution of  $(P_\lambda)$ . A solution  $u$  of  $(P_\lambda)$  is said to be *nontrivial and non-negative* if it satisfies  $u \geq 0$  on  $\overline{\Omega}$  and  $u \not\equiv 0$ , whereas it is said to be *positive* if it satisfies  $u > 0$  on  $\overline{\Omega}$ .

If  $\lambda > 0$  and  $a, b$  are positive on some non-empty open subset of  $\Omega$  then  $f_\lambda(x, s) = \lambda b(x)|s|^{q-2}s + a(x)|s|^{p-2}s$  belongs to the class of *concave-convex* type nonlinearities. Since the work of Ambrosetti, Brezis and Cerami [3], this class of problems has been widely investigated, mostly for Dirichlet boundary conditions. In [3] the authors proved the existence of  $\Lambda > 0$  such that the problem

$$\begin{cases} -\Delta u = \lambda|u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

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has a minimal positive solution  $u_\lambda$  for  $0 < \lambda < \Lambda$ , at least one positive weak solution for  $\lambda = \Lambda$ , and no positive solution for  $\lambda > \Lambda$  [3, Theorem 2.1]. Moreover, if  $p \leq \frac{2N}{N-2}$  when  $N \geq 3$  then (1.1) has a second positive solution  $v_\lambda > u_\lambda$  for  $\lambda < \Lambda$  [3, Theorem 2.3]. It was also proved that  $u_\lambda$  is the only positive solution of (1.1) which converges to 0 in  $C(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$  [3, Theorem 2.2]. Most of the previous results were extended by De Figueiredo, Gossez, and Ubilla [14] to a larger class of concave-convex type problems, whose prototype is the analogue of  $(P_\lambda)$  for Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta u = \lambda b(x)|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Here  $b \geq 0$ ,  $b \not\equiv 0$  and  $a$  may change sign. For other works dealing with non-negative solutions of indefinite concave-convex problems under Dirichlet boundary conditions we refer to [12, 23, 28].

Several differences between  $(P_\lambda)$  and (1.2) may be observed. The most evident one arises in the definite case  $a, b \geq 0$ , with  $a, b \not\equiv 0$ . It is known from [13, 14] that in this case (1.2) has a nontrivial non-negative solution for some  $\lambda > 0$ . This result no longer holds for  $(P_\lambda)$ . As a matter of fact, if  $u$  is a non-negative solution of  $(P_\lambda)$  then a simple integration provides

$$\int_{\Omega} (\lambda b(x)u^{q-1} + a(x)u^{p-1}) = 0,$$

so that  $u \equiv 0$  if  $\lambda > 0$ .

The first purpose of this work is to obtain conditions on  $a$  and  $b$  which guarantee the existence of a nontrivial non-negative solution of  $(P_\lambda)$  for some  $\lambda > 0$ . In particular, we shall obtain two nontrivial non-negative solutions  $u_{1,\lambda}, u_{2,\lambda}$  for  $\lambda > 0$  sufficiently small. At this point further differences between  $(P_\lambda)$  and (1.2) may be pointed out. Unlike [3, Theorem 2.2], we shall see that in some cases we have  $u_{1,\lambda}, u_{2,\lambda} \rightarrow 0$  in  $C(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$  (see Theorem 1.1). Furthermore, in contrast with [3, 14], the second solution  $u_{2,\lambda}$  may be obtained without the condition  $p \leq \frac{2N}{N-2}$  when  $N \geq 3$  (see Remark 1.2).

To the best of our knowledge, very few works have been devoted to concave-convex problems under Neumann boundary conditions. Tarfulea [26] considered  $(P_\lambda)$  in the case  $b \equiv 1$ , proving that  $\int_{\Omega} a < 0$  is a necessary and sufficient condition for the existence of a positive solution. Making use of the sub-supersolutions method, the author proved the existence of  $\Lambda > 0$  such that problem  $(P_\lambda)$  has at least one positive solution for  $\lambda < \Lambda$  which converges to 0 in  $L^\infty(\Omega)$  as  $\lambda \rightarrow 0^+$ , and no positive solution for  $\lambda > \Lambda$ .

Garcia-Azorero, Peral, and Rossi [15] dealt with the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

By means of a variational approach, they proved that if  $1 < q < 2$  and  $p = \frac{2N}{N-2}$  when  $N > 2$  then there exists  $\Lambda_1 > 0$  such that (1.3) has at least two positive solutions for  $\lambda < \Lambda_1$ , at least one positive solution for  $\lambda = \Lambda_1$ , and no positive solution for  $\lambda > \Lambda_1$ .

In [1], Alama investigated the problem

$$\begin{cases} -\Delta u = \mu u + b(x)u^{q-1} + \gamma u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where  $\mu \in \mathbb{R}$  and  $\gamma > 0$ . Note that when  $\mu = 0$  this problem can be reduced to  $(P_\lambda)$  by a suitable rescaling. A special difficulty in this problem is the possible existence of *dead core* solutions when  $b$  changes sign. Using variational, bifurcation, and sub-supersolutions

techniques, the author proved existence, non-existence and multiplicity results for non-negative solutions in accordance with  $\gamma$  and  $\mu$ . Moreover, these solutions are shown to be positive in the set where  $b > 0$ . However, the author did not discuss the structure of the non-negative solutions set when  $\mu = 0$ .

The second and main purpose of this article is to investigate the existence of a subcontinuum of non-negative solutions of  $(P_\lambda)$ . Some works have been devoted to this issue in the context of concave-convex nonlinearities. In [18], Korman proved that if  $\Omega$  is a ball in  $\mathbb{R}^N$  then there exists  $\lambda_0 > 0$  such that the problem

$$\begin{cases} -\Delta u = \lambda (|u|^{p-2}u + |u|^{q-2}u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

has exactly two positive solutions for  $\lambda < \lambda_0$ , one positive solution for  $\lambda = \lambda_0$  and no positive solution for  $\lambda > \lambda_0$ . In addition, he proved that for  $\lambda \leq \lambda_0$  the positive solutions lie on a single smooth solution curve and described the behavior of this curve with respect to  $\lambda$ . In [19] he extended these results to a problem with a non-autonomous concave-convex nonlinearity. Delgado and Suárez [12] considered the problem

$$\begin{cases} \mathcal{L}u = \lambda |u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

where  $\mathcal{L}$  is a second order uniformly elliptic operator not necessarily self-adjoint and  $a$  changes sign. They proved the existence of a unbounded subcontinuum of non-negative solutions emanating supercritically from  $(\lambda, u) = (0, 0)$ . To the best of our knowledge, no results on the existence of a subcontinuum of non-negative solutions for  $(P_\lambda)$  are known when  $b$  changes sign.

Based on the asymptotic analysis of  $u_{1,\lambda}, u_{2,\lambda}$  as  $\lambda \rightarrow 0^+$ , we shall prove in some cases the existence of a loop type subcontinuum (see Theorem 1.6) in the non-negative solutions set of  $(P_\lambda)$ . This kind of continuum has been investigated by López-Gómez and Molina-Meyer in [21] and Brown in [6] for problems involving nonlinearities that are  $C^1$  at  $u = 0$ , which is not the case for  $(P_\lambda)$ . For that same reason, the standard global bifurcation theory proposed by Rabinowitz [24] (see also López-Gómez [20]) does not apply to  $(P_\lambda)$  in a straightforward way. We shall overcome this difficulty using a regularization procedure that will be described later. Several works have made a direct use of the global bifurcation theory. We refer to Hess and Kato [17] for a problem with a non self-adjoint operator, to Blat and Brown [5] for a class of nonlinear elliptic systems, to López-Gómez and Molina-Meyer [21] for a study of isolas or compact solution components, to Cantrell and Cosner [9] for diffusive logistic equations from Mathematical Biology, and to Umezū [27] and Cano-Casanova [8] for nonlinear boundary conditions.

Note that if  $b \geq 0$  then, by the strong maximum principle and the boundary point lemma, nontrivial non-negative solutions of  $(P_\lambda)$  are positive solutions. On the other hand, it is known that if  $b^- \not\equiv 0$  then *dead core* solutions may arise [4], which makes delicate the study of the non-negative solutions set of  $(P_\lambda)$ , as shown in [1]. For instance, when  $b$  changes sign the existence of a minimal non-negative solution for  $\lambda > 0$  small is still unknown. Furthermore, when  $a \geq 0$  and  $b$  changes sign, it is not known whether the condition  $\int_\Omega b \geq 0$  provides non-existence of nontrivial non-negative solutions of  $(P_\lambda)$  for  $\lambda > 0$ .

In our existence results we shall also be concerned with stability properties of positive solutions of  $(P_\lambda)$ . Let us recall that a positive solution  $u$  of  $(P_\lambda)$  is said to be *asymptotically stable* (respect. *unstable*) if  $\gamma_1(\lambda, u) > 0$  (respect.  $< 0$ ), where  $\gamma_1(\lambda, u)$  is the first eigenvalue

of the linearized problem at  $u$ , namely,

$$\begin{cases} -\Delta\phi = (p-1)a(x)u^{p-2}\phi + \lambda(q-1)b(x)u^{q-2}\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.7)$$

In addition,  $u$  is said to be *weakly stable* if  $\gamma_1(\lambda, u) \geq 0$ .

Throughout this article, we consider the following sets:

$$\Omega_{\pm}^a = \{x \in \Omega : a(x) \gtrless 0\}, \quad \Omega_0^a = \{x \in \Omega : a(x) = 0\}, \quad \Omega_{\pm}^b = \{x \in \Omega : b(x) \gtrless 0\}.$$

Our main existence results for  $\lambda > 0$  shall be obtained under the condition

$$\int_{\Omega} a < 0 \quad \text{or} \quad \int_{\Omega} b < 0. \quad (1.8)$$

If either  $\int_{\Omega} a < 0 \leq \int_{\Omega} b$  or  $\int_{\Omega} a > 0 \geq \int_{\Omega} b$  then we set

$$c^* = \left( \frac{-\int_{\Omega} b}{\int_{\Omega} a} \right)^{\frac{1}{p-q}}. \quad (1.9)$$

We are now in position to state out main results.

First we follow a variational approach to show that  $(P_{\lambda})$  has two nontrivial non-negative solutions for  $\lambda > 0$  small if (1.8) holds and  $\Omega_+^a, \Omega_+^b \neq \emptyset$ . This approach also provides us with the asymptotic profiles of these solutions as  $\lambda \rightarrow 0^+$ :

**Theorem 1.1.** *Assume (1.8) and  $p < \frac{2N}{N-2}$  if  $N \geq 3$ . Then there exists  $\lambda_0 > 0$  such that:*

- (1) *If  $\Omega_+^b \neq \emptyset$  then  $(P_{\lambda})$  has a nontrivial non-negative solution  $u_{1,\lambda}$  for  $0 < \lambda < \lambda_0$ . Moreover there holds  $u_{1,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . More precisely:*

- (a) *If, in addition,  $\int_{\Omega} b < 0$  and  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $\lambda_n^{-\frac{1}{2-q}} u_{1,\lambda_n} \rightarrow w_0$  in  $C^2(\overline{\Omega})$  as  $n \rightarrow \infty$ , where  $w_0$  is a nontrivial non-negative ground state solution of*

$$\begin{cases} -\Delta w = b(x)|w|^{q-2}w & \text{in } \Omega, \\ \frac{\partial w}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.10)$$

- (b) *If, in addition,  $\int_{\Omega} a < 0 \leq \int_{\Omega} b$  then  $\lambda^{-\frac{1}{p-q}} u_{1,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . In particular, if  $\int_{\Omega} a < 0 < \int_{\Omega} b$  then  $u_{1,\lambda}$  is an asymptotically stable positive solution of  $(P_{\lambda})$  for  $\lambda > 0$  sufficiently small.*

- (2) *If  $\Omega_+^a \neq \emptyset$  then  $(P_{\lambda})$  has a nontrivial non-negative solution  $u_{2,\lambda}$  for  $0 < \lambda < \lambda_0$ . Moreover there holds:*

- (a) *If, in addition,  $\int_{\Omega} a > 0 > \int_{\Omega} b$  then  $u_{2,\lambda} \rightarrow 0$  and  $\lambda^{-\frac{1}{p-q}} u_{2,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . In particular,  $u_{2,\lambda}$  is a unstable positive solution of  $(P_{\lambda})$  for  $\lambda > 0$  sufficiently small.*

- (b) *If, in addition,  $\int_{\Omega} a = 0 > \int_{\Omega} b$  then  $u_{2,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ .*

- (c) *If, in addition,  $\int_{\Omega} a < 0$  and  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $u_{2,\lambda_n} \rightarrow u_{2,0}$  in  $C^2(\overline{\Omega})$  as  $n \rightarrow \infty$ , where  $u_{2,0}$  is a positive ground state solution of*

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.11)$$

*In particular,  $u_{2,\lambda}$  is a unstable positive solution of  $(P_{\lambda})$  for  $\lambda > 0$  sufficiently small.*

**Remark 1.2.**

- (1) Except for (1)(a), 1(b) with  $\int_{\Omega} b = 0$ , 2(b) and (2)(c), Theorem 1.1 remains true without the condition  $p < \frac{2N}{N-2}$  if  $N \geq 3$ . In the case  $\Omega_+^b \neq \emptyset$  and  $\int_{\Omega} b < 0$  a solution having similar features as  $u_{1,\lambda}$  may be obtained by the sub-supersolutions method. Note that in contrast with the case of Dirichlet boundary conditions, obtaining a strict supersolution for  $(P_{\lambda})$  is not an easy task. We shall use the asymptotic profile of  $u_{1,\lambda}$  provided by Theorem 1.1 (1)(a) to obtain such a supersolution, cf. Proposition 3.1. In the case  $\int_{\Omega} a < 0 < \int_{\Omega} b$  we shall use the Lyapunov-Schmidt reduction method to obtain a positive solution  $u_{\lambda}$  such that  $\lambda^{-\frac{1}{p-q}} u_{\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . The same procedure can be applied in the case  $\int_{\Omega} a > 0 > \int_{\Omega} b$ , cf. Remark 5.6.
- (2) When  $\Omega_+^b$  is a non-empty subdomain of  $\Omega$  and  $\int_{\Omega} b < 0$ , we deduce from Theorem 1.1 (1)(a) that for any subset  $D$  satisfying  $\overline{D} \subset \Omega_+^b$  there exist  $\overline{\lambda}, \overline{c} > 0$  such that  $\inf_D u_{1,\lambda} \geq \overline{c}$  for  $\lambda \in (0, \overline{\lambda})$ . This result comes from the fact that  $w_0 > 0$  in  $\Omega_+^b$ .

From Theorem 1.1 we infer in particular some positivity and ordering properties for  $u_{1,\lambda}$  and  $u_{2,\lambda}$  (cf. Remark 2.13 (1)):

**Corollary 1.3.** *Assume  $p < \frac{2N}{N-2}$  if  $N \geq 3$ . Let  $u_{1,\lambda}$  and  $u_{2,\lambda}$  be provided by Theorem 1.1.*

- (1) *If  $\Omega_+^a \neq \emptyset$  and  $\int_{\Omega} a < 0 < \int_{\Omega} b$  then there exists  $\lambda^* > 0$  such that  $u_{2,\lambda} > u_{1,\lambda} > 0$  on  $\overline{\Omega}$  for  $0 < \lambda < \lambda^*$ .*
- (2) *If  $\Omega_+^b \neq \emptyset$  and  $\int_{\Omega} a > 0 > \int_{\Omega} b$  then there exists  $\lambda^* > 0$  such that  $u_{2,\lambda} > u_{1,\lambda} \geq 0$  on  $\overline{\Omega}$  for  $0 < \lambda < \lambda^*$ .*

As for non-existence of nontrivial non-negative solutions of  $(P_{\lambda})$ , we have the following result:

**Theorem 1.4.**

- (1) *Let  $\lambda > 0$ . Then the following two assertions hold:*
- (a) *Assume  $b \geq 0$  and  $\int_{\Omega} a \geq 0$ . Then  $(P_{\lambda})$  has no nontrivial non-negative solution.*
- (b) *Assume that  $b$  changes sign,  $\Omega_+^b$  is a subdomain of  $\Omega$ , and  $\Omega_-^b = \Omega \setminus \overline{\Omega_+^b}$ . If  $a \geq 0$  and  $\int_{\Omega} b \geq 0$  then  $(P_{\lambda})$  has no non-negative solution taking positive values somewhere in  $\Omega_+^b$ .*
- (2) *Assume  $\Omega_+^a \cap \Omega_+^b \neq \emptyset$ . Then there exists  $\overline{\lambda} > 0$  such that  $(P_{\lambda})$  has no nontrivial non-negative solution for  $\lambda > \overline{\lambda}$ .*

**Remark 1.5.** Theorem 1.4 holds true for  $\lambda < 0$  with  $b$  replaced by  $-b$ . Indeed, it suffices to look at the equation in  $(P_{\lambda})$  as  $-\Delta u = (-\lambda)(-b(x))|u|^{q-2}u + a(x)|u|^{p-2}u$ .

We consider then structure of the non-negative solutions set of  $(P_{\lambda})$ . Under the condition

$$\Omega_+^a \neq \emptyset, \quad \Omega_+^b \neq \emptyset, \quad \int_{\Omega} b \leq 0 \quad \text{and} \quad \int_{\Omega} a < 0, \quad (1.12)$$

Theorem 1.1 asserts that  $u_{1,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ , and if  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $u_{2,\lambda_n} \rightarrow u_{2,0}$  in  $C^2(\overline{\Omega})$ , where  $u_{2,0}$  is a positive solution of (1.11). In addition, this result does not depend on the sign of  $\int_{\Omega} b$ . As a consequence we may also infer the existence of two nontrivial non-negative solutions  $v_{1,\lambda}$  and  $v_{2,\lambda}$  for  $\lambda < 0$  sufficiently small. These solutions satisfy  $v_{1,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ , and if  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $v_{2,\lambda_n} \rightarrow v_{2,0}$  in  $C^2(\overline{\Omega})$ , where  $v_{2,0}$  is a positive solution of (1.11). One may then ask if these solutions lie on a loop type subcontinuum of non-negative solutions of  $(P_{\lambda})$ .

We shall investigate this question by considering a regularized version of  $(P_\lambda)$ , namely,

$$(P_{\lambda,\epsilon}) \quad \begin{cases} -\Delta u = a(x)|u|^{p-2}u + \lambda(b(x) - \epsilon)|u + \epsilon|^{q-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\lambda \in \mathbb{R}$  and  $\epsilon > 0$ . We may then look at  $(P_\lambda)$  as the limit problem of  $(P_{\lambda,\epsilon})$  when  $\epsilon \rightarrow 0^+$ . This procedure has been already used in [25], where a regularized version of a nonlinear boundary condition is studied. Note that the mapping  $t \mapsto |t + \epsilon|^{q-2}t$  is analytic at  $t = 0$  and any nontrivial non-negative solution of  $(P_{\lambda,\epsilon})$  is positive on  $\overline{\Omega}$ . The unilateral global bifurcation theorem by Rabinowitz [24, Theorem 1.27] (see also López-Gómez [20, Theorem 6.4.3]) may then be applied to  $(P_{\lambda,\epsilon})$ . To this end we consider its linearized problem at  $u = 0$ :

$$\begin{cases} -\Delta \varphi = \lambda(b - \epsilon)\epsilon^{q-2}\varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.13)$$

Under the condition

$$\Omega_+^{b-\epsilon} \neq \emptyset \quad \text{and} \quad \int_{\Omega} b \leq 0, \quad (1.14)$$

this problem has exactly two principal eigenvalues  $\lambda = 0$  and  $\lambda = \lambda_\epsilon > 0$ , which are both simple. We use the unilateral global bifurcation theory to obtain two subcontinua  $\mathcal{C}_0 = \mathcal{C}_0(\epsilon)$ ,  $\mathcal{C}_1 = \mathcal{C}_1(\epsilon)$  of positive solutions of  $(P_{\lambda,\epsilon})$  bifurcating from  $(0, 0)$  and  $(\lambda_\epsilon, 0)$ , respectively. Moreover, we analyse the local nature of these subcontinua near the bifurcation points (Theorem 5.1). We turn then to the study of the global nature of  $\mathcal{C}_0(\epsilon)$ ,  $\mathcal{C}_1(\epsilon)$  and their limiting nature as  $\epsilon \rightarrow 0^+$ . First we show that positive solutions  $(\lambda, u)$  of  $(P_{\lambda,\epsilon})$  are *a priori* bounded in  $\mathbb{R} \times C(\overline{\Omega})$  if the following conditions are assumed, where we assume  $(H_2)$  following Amann and López-Gómez [2]:

$(H_0)$  There exist balls  $B_1, B_2$  such that  $\overline{B_1}, \overline{B_2} \subset \Omega$ , and

$$\begin{cases} a \geq 0, & a \not\equiv 0 \quad \text{and } b > 0 \text{ on } \overline{B_1}, \\ a \geq 0, & a \not\equiv 0 \quad \text{and } b < 0 \text{ on } \overline{B_2}, \end{cases}$$

$(H_1)$   $\Omega_\pm^a$  are subdomains of  $\Omega$  with smooth boundary and satisfy  $\overline{\Omega_+^a} \subset \Omega$ ,  $\overline{\Omega_+^a} \cup \Omega_-^a = \Omega$ .

$(H_2)$  Under  $(H_1)$  there exist a function  $\alpha^+$  which is continuous, positive, and bounded away from zero in a tubular neighborhood of  $\partial\Omega_+^a$  in  $\Omega_+^a$  and  $\gamma > 0$  such that

$$a^+(x) = \alpha^+(x) \text{dist}(x, \partial\Omega_+^a)^\gamma,$$

where  $\text{dist}(x, A)$  denotes the distance function to a set  $A$ . Moreover, we assume that

$$2 < p < \min \left\{ \frac{2N}{N-2}, \frac{2N+\gamma}{N-1} \right\} \quad \text{if } N > 2.$$

Based on these *a priori* bounds and the global properties of  $\mathcal{C}_0$  and  $\mathcal{C}_1$ , we infer that these subcontinua are both bounded, and consequently must coincide, i.e.  $\mathcal{C}_0 = \mathcal{C}_1 = \mathcal{C}_*$  (Theorem 6.7). Thus  $(P_{\lambda,\epsilon})$  has a bounded subcontinuum of positive solutions going from  $(0, 0)$  to  $(\lambda_\epsilon, 0)$ , see Figure 2. We consider then the limiting profiles of  $\mathcal{C}_0$  and  $\mathcal{C}_1$  as  $\epsilon \rightarrow 0^+$  by means of Whyburn's topological method [29]. Here *a priori* bounds from below for positive solutions of  $(P_{\lambda,\epsilon})$  with  $\lambda = 0$  (Lemma 6.8) and the fact that bifurcation from zero does not

occur for  $(P_\lambda)$  at any  $\lambda \neq 0$  (Proposition 6.10) play an important role. The latter fact is verified under the condition

$(H_3)$   $\Omega_+^b$  and  $\Omega_-^b$  are both subdomains of  $\Omega$ .

Combining the previous results, we establish:

**Theorem 1.6.** *Assume (1.12). If  $(H_0)$ ,  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  are satisfied then  $(P_\lambda)$  has a loop type subcontinuum (non-empty, closed and connected component)  $\mathcal{C}$  of nontrivial non-negative solutions bifurcating at  $(0, 0)$ , which joins  $(0, 0)$  to itself. Moreover:*

- (1)  $\mathcal{C}$  is non-trivial, i.e.  $\mathcal{C} \neq \{(0, 0)\}$ .
- (2) The only trivial solution contained in  $\mathcal{C}$  is  $(\lambda, u) = (0, 0)$ , i.e.  $\mathcal{C}$  does not contain any point  $(\lambda, 0)$  with  $\lambda \neq 0$ .
- (3) There exists  $\delta > 0$  such that  $\mathcal{C}$  does not contain any positive solution  $u$  of (1.11) satisfying  $\|u\|_{C(\overline{\Omega})} \leq \delta$ .

Figure 3 illustrates the subcontinuum provided by Theorems 1.6.

**Remark 1.7.** An example of  $(a, b)$  satisfying conditions  $(H_0)$ ,  $(H_1)$  and  $(H_3)$  can be constructed as in Figure 1.

Finally, let us mention that our regularization procedure described above can also be used to obtain subcontinua (non-necessarily of loop type) for a larger class of concave-convex type problems. We shall treat this issue in a forthcoming article.

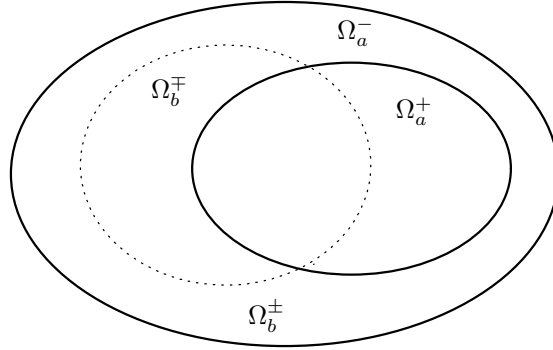
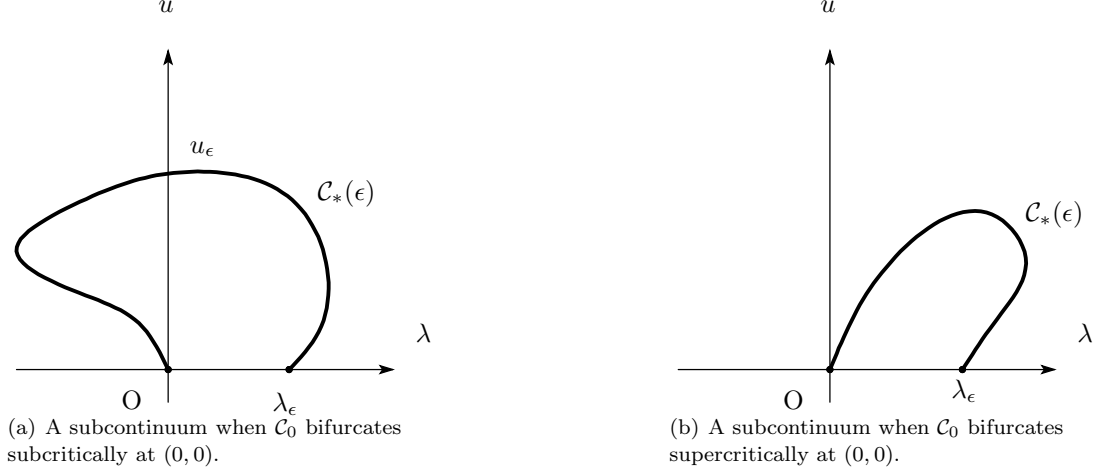
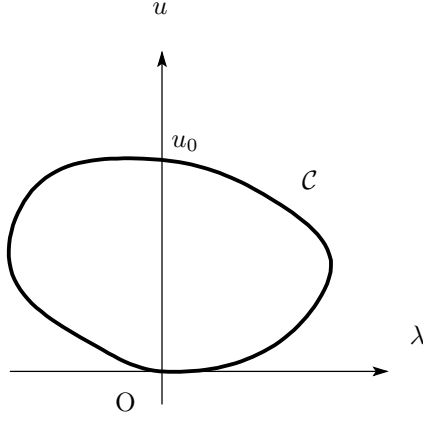


FIGURE 1. An example of  $(a, b)$  satisfying  $(H_0)$ ,  $(H_1)$  and  $(H_3)$ .

The outline of this article is the following: in Section 2 we follow a variational approach based on the Nehari manifold method to prove Theorem 1.1. In Section 3 we use the asymptotic profile of  $u_{1,\lambda}$  to obtain a nontrivial non-negative solution of  $(P_\lambda)$  for  $\lambda > 0$  small via the sub-supersolutions method. We also show that bifurcation from zero does not occur for  $(P_\lambda)$  at any  $\lambda > 0$ . Section 4 is devoted to the proof of Theorem 1.4. In Section 5 we carry out a bifurcation analysis for the regularized problem  $(P_{\lambda,\epsilon})$ . Finally, in Section 6 we prove Theorem 1.6.

**1.1. Notation.** Throughout this article we use the following notations and conventions:

- The infimum of an empty set is assumed to be  $\infty$ .
- Unless otherwise stated, for any  $f \in L^1(\Omega)$  the integral  $\int_\Omega f$  is considered with respect to the Lebesgue measure, whereas for any  $g \in L^1(\partial\Omega)$  the integral  $\int_{\partial\Omega} g$  is considered with respect to the surface measure.

FIGURE 2. Bounded subcontinua of  $(P_{\lambda, \epsilon})$  or  $(P_{\lambda, \epsilon}')$ .FIGURE 3. A loop type subcontinuum of  $(P_\lambda)$  when (1.12) holds.

- For  $r \geq 1$  the Lebesgue norm in  $L^r(\Omega)$  will be denoted by  $\|\cdot\|_r$  and the usual norm of  $H^1(\Omega)$  by  $\|\cdot\|$ .
- The strong and weak convergence are denoted by  $\rightarrow$  and  $\rightharpoonup$ , respectively.
- The positive and negative parts of a function  $u$  are defined by  $u^\pm := \max\{\pm u, 0\}$ .
- If  $U \subset \mathbb{R}^N$  then we denote the closure of  $U$  by  $\overline{U}$  and the interior of  $U$  by  $\text{int } U$ .
- The support of a measurable function  $f$  is denoted by  $\text{supp } f$ .

## 2. THE VARIATIONAL APPROACH

Throughout this section we assume that  $p < \frac{2N}{N-2}$ . We associate to  $(P_\lambda)$  the  $\mathcal{C}^1$  functional  $I_\lambda$  defined on  $X$  by

$$I_\lambda(u) := \frac{1}{2}E(u) - \frac{1}{p}A(u) - \frac{\lambda}{q}B(u),$$



where

$$E(u) = \int_{\Omega} |\nabla u|^2, \quad A(u) = \int_{\Omega} a(x)|u|^p, \quad \text{and} \quad B(u) = \int_{\Omega} b(x)|u|^q.$$

Let us recall that  $X = H^1(\Omega)$  is equipped with the usual norm  $\|u\| = [\int_{\Omega} (|\nabla u|^2 + u^2)]^{\frac{1}{2}}$ . Critical points of  $I_{\lambda}$  are weak solutions of  $(P_{\lambda})$ , which are also classical solutions by standard regularity. In the sequel we shall consider the following useful subsets of  $X$ :

$$\begin{aligned} E^+ &= \{u \in X : E(u) > 0\}, \\ A^{\pm} &= \{u \in X : A(u) \gtrless 0\}, \quad A_0 = \{u \in X : A(u) = 0\}, \quad A_0^{\pm} = A^{\pm} \cup A_0. \\ B^{\pm} &= \{u \in X : B(u) \gtrless 0\}, \quad B_0 = \{u \in X : B(u) = 0\}, \quad B_0^{\pm} = B^{\pm} \cup B_0. \end{aligned}$$

The next result will be used repeatedly in this section:

**Lemma 2.1.**

- (1) If  $(u_n)$  is a sequence such that  $u_n \rightharpoonup u_0$  in  $X$  and  $\limsup_n E(u_n) \leq 0$  then  $u_0$  is a constant and  $u_n \rightarrow u_0$  in  $X$ .
- (2) Assume  $\int_{\Omega} a < 0$  (respect.  $\int_{\Omega} b < 0$ ). If  $v \not\equiv 0$  and  $v \in A_0^+$  (respect.  $v \in B_0^+$ ) then  $v$  is not a constant.

*Proof.*

- (1) Since  $u_n \rightharpoonup u_0$  in  $X$  and  $E$  is weakly lower semicontinuous, we have

$$0 \leq E(u_0) \leq \liminf E(u_n) \leq 0.$$

Hence  $E(u_0) = 0$ , which implies that  $u_0$  is a constant. Moreover  $E(u_n) \rightarrow E(u_0)$ . Since  $u_n \rightarrow u_0$  in  $L^2(\Omega)$  we deduce that  $u_n \rightarrow u_0$  in  $X$ .

- (2) If  $v \in A_0^+$  is a non-zero constant then  $0 \leq A(v_0) = |v_0|^p \int_{\Omega} a < 0$ , which is a contradiction.

□

The Nehari manifold associated to  $I_{\lambda}$  is given by

$$N_{\lambda} := \{u \in X \setminus \{0\} : \langle I'_{\lambda}(u), u \rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = A(u) + \lambda B(u)\}.$$

We shall use the splitting

$$N_{\lambda} = N_{\lambda}^+ \cup N_{\lambda}^- \cup N_{\lambda}^0,$$

where

$$\begin{aligned} N_{\lambda}^{\pm} &:= \{u \in N_{\lambda} : \langle J'_{\lambda}(u), u \rangle \gtrless 0\} = \left\{ u \in N_{\lambda} : E(u) \lesseqgtr \lambda \frac{p-q}{p-2} B(u) \right\} \\ &= \left\{ u \in N_{\lambda} : E(u) \gtrless \frac{p-q}{2-q} A(u) \right\}, \end{aligned}$$

and

$$N_{\lambda}^0 = \{u \in N_{\lambda} : \langle J'_{\lambda}(u), u \rangle = 0\}.$$

Note that any nontrivial solution of  $(P_{\lambda})$  belongs to  $N_{\lambda}$ . Furthermore, it follows from the implicit function theorem that  $N_{\lambda} \setminus N_{\lambda}^0$  is a  $C^1$  manifold and every critical point of the restriction of  $I_{\lambda}$  to this manifold is a critical point of  $I_{\lambda}$  (see for instance [7, Theorem 2.3]), and therefore a solution of  $(P_{\lambda})$ .

**Remark 2.2.** Note that any positive solution of  $(P_\lambda)$  belonging to  $N_\lambda^-$  is unstable. Indeed, if  $u \in N_\lambda^-$  then

$$E(u) - (p-1)A(u) - \lambda(q-1)B(u) = (2-q)E(u) - (p-q)A(u) < 0.$$

It follows that

$$\gamma_1(\lambda, u) = \inf \left\{ \int_{\Omega} (|\nabla \phi|^2 - (p-1)a(x)u^{p-2}\phi^2 - \lambda(q-1)b(x)u^{q-2}\phi^2) : \|\phi\|_2 = 1 \right\} < 0.$$

To analyse the structure of  $N_\lambda^\pm$ , we consider the fibering maps corresponding to  $I_\lambda$ , which are set, for  $u \neq 0$ , as follows:

$$j_u(t) := I_\lambda(tu) = \frac{t^2}{2}E(u) - \frac{t^p}{p}A(u) - \lambda \frac{t^q}{q}B(u), \quad t > 0.$$

It is easy to see that

$$j'_u(1) = 0 \leq j''_u(1) \iff u \in N_\lambda^\pm,$$

and more generally,

$$j'_u(t) = 0 \leq j''_u(t) \iff tu \in N_\lambda^\pm.$$

Having this characterisation in mind, we look for conditions under which  $j_u$  has a critical point. Set

$$i_u(t) := t^{-q}j_u(t) = \frac{t^{2-q}}{2}E(u) - \frac{t^{p-q}}{p}A(u) - \lambda B(u), \quad t > 0.$$

Let  $u \in E^+ \cap A^+ \cap B^+$ . Then  $i_u$  has a global maximum  $i_u(t^*)$  at some  $t^* > 0$ , and moreover,  $t^*$  is unique. If  $i_u(t^*) > 0$ , then  $j_u$  has a global maximum which is positive and a local minimum which is negative. Moreover, these are the only critical points of  $j_u$ .

We shall require a condition on  $\lambda$  that provides  $i_u(t^*) > 0$ . Note that

$$i'_u(t) = \frac{2-q}{2}t^{1-q}E(u) - \frac{p-q}{p}t^{p-q-1}A(u) = 0$$

if and only if

$$t = t^* := \left( \frac{p(2-q)E(u)}{2(p-q)A(u)} \right)^{\frac{1}{p-2}}.$$

Moreover

$$i_u(t^*) = \frac{p-2}{2(p-q)} \left( \frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-2}} \frac{E(u)^{\frac{p-q}{p-2}}}{A(u)^{\frac{2-q}{p-2}}} - \frac{\lambda}{q}B(u) > 0$$

if and only if

$$0 < \lambda < C_{pq} \frac{E(u)^{\frac{p-q}{p-2}}}{B(u)A(u)^{\frac{2-q}{p-2}}}, \quad (2.1)$$

where  $C_{pq} = \left( \frac{q(p-2)}{2(p-q)} \right) \left( \frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-2}}$ . Note that  $F(u) = \frac{E(u)^{\frac{p-q}{p-2}}}{B(u)A(u)^{\frac{2-q}{p-2}}}$  satisfies  $F(tu) = F(u)$  for  $t > 0$ , i.e.  $F$  is homogeneous of order 0.

We introduce now

$$\lambda_0 = \inf \{ E(u)^{\frac{p-q}{p-2}} : u \in E^+ \cap A^+ \cap B^+, C_{pq}^{-1}B(u)A(u)^{\frac{2-q}{p-2}} = 1 \}. \quad (2.2)$$

Note that if  $E^+ \cap A^+ \cap B^+ = \emptyset$  then  $\lambda_0 = \infty$ . We deduce then the following result, which provides sufficient conditions for the existence of critical points of  $j_u$ :

**Proposition 2.3.** *Assume (1.8). Then  $\lambda_0 > 0$  and, for  $0 < \lambda < \lambda_0$ , there holds:*

- (1) If either  $u \in A^+ \cap B^-$  or  $u \in E^+ \cap A^+ \cap B_0$  then,  $j_u$  has a positive global maximum at some  $t_1 > 0$ , i.e.  $j'_u(t_1) = 0 > j''_u(t_1)$  and  $j_u(t) < j_u(t_1)$  for  $t \neq t_1$ . Moreover,  $t_1$  is the unique critical point of  $j_u$ .
- (2) If either  $u \in A^- \cap B^+$  or  $E^+ \cap A_0 \cap B^+$  then  $j_u$  has a negative global minimum at some  $t_1 > 0$ , i.e.  $j'_u(t_1) = 0 < j''_u(t_1)$  and  $j_u(t) > j_u(t_1)$  for  $t \neq t_1$ . Moreover,  $t_1$  is the unique critical point of  $j_u$ .
- (3) If  $u \in E^+ \cap A^+ \cap B^+$  then  $j_u$  has a negative local minimum at  $t_1 > 0$  and a positive global maximum at  $t_2 > t_1$ . Furthermore  $t_1$  and  $t_2$  are the only critical points of  $j_u$ .

*Proof.* First, we show that  $\lambda_0 > 0$ . Assume  $\lambda_0 = 0$ , so that we can choose  $u_n \in E^+ \cap A^+ \cap B^+$  satisfying

$$E(u_n) \rightarrow 0, \quad \text{and} \quad C_{pq}^{-1} B(u_n) A(u_n)^{\frac{2-q}{p-2}} = 1.$$

If  $(u_n)$  is bounded in  $X$  then we may assume that  $u_n \rightharpoonup u_0$  for some  $u_0 \in X$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows from Lemma 2.1(1) that  $u_0$  is a constant and  $u_n \rightarrow u_0$  in  $X$ . From  $u_n \in A^+ \cap B^+$  we deduce that  $u_0 \in A_0^+ \cap B_0^+$ . In addition, there holds

$$C_{pq}^{-1} B(u_0) A(u_0)^{\frac{2-q}{p-2}} = 1,$$

so that  $u_0 \not\equiv 0$ . From Lemma 2.1 we get a contradiction.

Let us assume now that  $\|u_n\| \rightarrow \infty$ . Set  $v_n = \frac{u_n}{\|u_n\|}$ , so that  $\|v_n\| = 1$ . We may assume that  $v_n \rightharpoonup v_0$  and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$ . Since  $E(v_n) \rightarrow 0$  and  $v_n \in A^+ \cap B^+$ , we have  $v_n \rightarrow v_0$  in  $X$ ,  $v_0$  is a constant, and  $v_0 \in A_0^+ \cap B_0^+$ . In particular,  $\|v_0\| = 1$ , i.e.  $v_0 \not\equiv 0$ . Lemma 2.1 provides again a contradiction.

Now, if  $u \in A^+ \cap B^-$  or  $u \in E^+ \cap A^+ \cap B_0$  then it is clear that  $j_u$  has a unique critical point, which is a global maximum point. In a similar way, if  $u \in A^- \cap B^+$  then  $j_u$  has a unique critical point, which is a global minimum point. Finally, if  $u \in E^+ \cap A^+ \cap B^+$  then

$$\lambda_0 \leq C_{pq} \frac{E(u)^{\frac{p-q}{p-2}}}{B(u) A(u)^{\frac{2-q}{p-2}}}.$$

Thus, if  $0 < \lambda < \lambda_0$  then  $i_u(t^*) > 0$  from (2.1). This completes the proof.  $\square$

**Proposition 2.4.** Assume (1.8). Then, for  $0 < \lambda < \lambda_0$ , we have:

- (1)  $N_\lambda^0$  is empty.
- (2) If  $A^+ \cap B^+ \neq \emptyset$  then  $N_\lambda^+$  and  $N_\lambda^-$  are non-empty.
- (3) If  $A_0^- \cap B^+ \neq \emptyset$  then  $N_\lambda^+$  is non-empty.
- (4) If  $A^+ \cap B_0^- \neq \emptyset$  then  $N_\lambda^-$  is non-empty.

*Proof.* From Proposition 2.3 it follows that there is no  $t > 0$  such that  $j'_u(t) = j''_u(t) = 0$ , i.e.  $N_\lambda^0$  is empty. Let  $u_0 \in A^+ \cap B^+$ . Since  $\int_\Omega a < 0$  or  $\int_\Omega b < 0$ , from Lemma 2.1 it follows that  $u_0 \in E^+ \cap A^+ \cap B^+$ . By Proposition 2.3 we infer that for  $0 < \lambda < \lambda_0$  there are  $0 < t_1 < t_2$  such that  $t_1 u \in N_\lambda^+$  and  $t_2 u \in N_\lambda^-$ . Assertions (3) and (4) are straightforward.  $\square$

The following result provides some properties of  $N_\lambda^+$  and  $N_\lambda^-$ :

**Lemma 2.5.** Assume (1.8). Then we have:

- (1)  $N_\lambda^+ \subset B^+$  and  $N_\lambda^- \subset A^+$ .
- (2)  $N_\lambda^+$  is bounded in  $X$  for  $0 < \lambda < \lambda_0$ .

*Proof.*

- (1) Let  $u \in N_\lambda^+$ . Then  $0 \leq E(u) < \lambda \frac{p-q}{p-2} B(u)$ , i.e.  $u \in B^+$ . Now, if  $u \in N_\lambda^-$  then  $0 \leq E(u) < \frac{p-q}{2-q} A(u)$ , i.e.  $u \in A^+$ .
- (2) Assume  $(u_n) \subset N_\lambda^+$  and  $\|u_n\| \rightarrow \infty$ . Set  $v_n = \frac{u_n}{\|u_n\|}$ . It follows that  $\|v_n\| = 1$ , so we may assume that  $v_n \rightharpoonup v_0$ ,  $B(v_n)$  is bounded, and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$  (implying  $A(v) \rightarrow A(v_0)$ ). Since  $u_n \in N_\lambda^+$ , we see that

$$E(v_n) < \lambda \frac{p-q}{p-2} B(v_n) \|u_n\|^{q-2},$$

and thus  $\limsup E(v_n) \leq 0$ . Lemma 2.1(1) yields that  $v_0$  is a constant and  $v_n \rightarrow v_0$  in  $X$ . Consequently,  $\|v_0\| = 1$ , and  $v_0$  is a non-zero constant. On the other hand, since  $u_n \in N_\lambda^+$ , we have  $v_n \in N_\lambda^+$ , so  $v_n \in B^+$ . It follows that  $v_0 \in B_0^+$ . Finally, from

$$E(u_n) = \lambda B(u_n) + A(u_n)$$

we deduce that  $A(v_n) \rightarrow 0$ , i.e.  $v_0 \in A_0$ . Therefore  $v_0 \in A_0 \cap B_0^+$ , which contradicts Lemma 2.1(2).  $\square$

**Proposition 2.6.** *Assume (1.8) and  $\Omega_+^b \neq \emptyset$ . Then there exists  $u_{1,\lambda} \geq 0$  such that  $I_\lambda(u_{1,\lambda}) = \min_{N_\lambda^+} I_\lambda$  for  $0 < \lambda < \lambda_0$ . In particular,  $u_{1,\lambda}$  is a nontrivial non-negative solution of  $(P_\lambda)$  for  $0 < \lambda < \lambda_0$ .*

*Proof.* Let  $0 < \lambda < \lambda_0$ . We consider a minimizing sequence  $(u_n) \subset N_\lambda^+$ , i.e.

$$I_\lambda(u_n) \rightarrow \inf_{N_\lambda^+} I_\lambda < 0.$$

Since  $(u_n)$  is bounded in  $X$ , we may assume that  $u_n \rightharpoonup u_0$  in  $X$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows that

$$I_\lambda(u_0) \leq \liminf I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda < 0,$$

so that  $u_0 \neq 0$ . Moreover, as  $u_n \in B^+$  we have  $u_0 \in B_0^+$ . We claim that  $u_0 \in B^+$ . Indeed, if  $u_0 \in B_0$  then, from

$$E(u_n) < \lambda \frac{p-q}{p-2} B(u_n)$$

we obtain  $E(u_0) = 0$ , i.e.  $u_0$  is a constant. So  $\int_\Omega b = 0$ , and consequently, by (1.8), we have  $\int_\Omega a < 0$ . It follows that  $j_{u_0}(t) = -\frac{1}{p} t^p |u_0|^p \int_\Omega a > 0$  for every  $t > 0$ , which contradicts  $j_{u_0}(1) = I_\lambda(u_0) < 0$ . Thus  $u_0 \in B^+$  and by Proposition 2.3 we have  $t_1 u_0 \in N_\lambda^+$  for some  $t_1 > 0$ . Assume  $u_n \not\rightarrow u_0$ . If  $1 < t_1$  then we have

$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf j_{u_n}(1) = \liminf I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \quad (2.3)$$

which is impossible. If  $t_1 \leq 1$  then  $j'_{u_n}(t_1) \leq 0$  for every  $n$ , so that  $j'_{u_0}(t_1) < \liminf j'_{u_n}(t_1) \leq 0$ , which is a contradiction. Therefore  $u_n \rightarrow u_0$ . Now, since  $u_n \rightarrow u_0$  we have  $j'_{u_0}(1) = 0 \leq j''_{u_0}(1)$ . But  $j''_{u_0}(1) = 0$  is impossible by Proposition 2.4(1). Thus  $u_0 \in N_\lambda^+$  and  $I_\lambda(u_0) = \inf_{N_\lambda^+} I_\lambda$ . We set  $u_{1,\lambda} = u_0$ .  $\square$

Next we obtain a second nontrivial non-negative solution of  $(P_\lambda)$ , which achieves  $\inf_{N_\lambda^-} I_\lambda$  for  $\lambda \in (0, \lambda_0)$ . The following result provides some properties of  $N_\lambda^-$ :

**Lemma 2.7.** *Assume (1.8). Then, for  $0 < \lambda < \lambda_0$ , we have  $I_\lambda(u) > 0$  for any  $u \in N_\lambda^-$ .*

*Proof.* Let  $u \in N_\lambda^-$ . By Lemma 2.5 we know that  $u \in A^+$ . If  $u \in B_0^+$  then  $u$  is non-constant, i.e.  $u \in E^+$ . Hence either  $u \in A^+ \cap B_0^+ \cap E^+$  or  $u \in A^+ \cap B^-$ . In both cases, by Proposition 2.3 we have that  $t = 1$  is the global maximum point of  $j_u$  and  $I_\lambda(u) = j_u(1) > 0$ .  $\square$

**Proposition 2.8.** *Assume (1.8) and  $\Omega_+^a \neq \emptyset$ . Then there exists  $u_{2,\lambda} \geq 0$  such that  $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda$  for  $\lambda \in (0, \lambda_0)$ . In particular,  $u_{2,\lambda}$  is a nontrivial non-negative solution of  $(P_\lambda)$  for  $\lambda \in (0, \lambda_0)$ .*

*Proof.* Since  $I_\lambda(u) > 0$  for  $u \in N_\lambda^-$ , we can choose  $u_n \in N_\lambda^-$  such that

$$I_\lambda(u_n) \rightarrow \inf_{N_\lambda^-} I_\lambda \geq 0.$$

We claim that  $(u_n)$  is bounded in  $X$ . Indeed, there exists  $C > 0$  such that  $I_\lambda(u_n) \leq C$ . Since  $u_n \in N_\lambda$ , we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) \leq C. \quad (2.4)$$

Assume  $\|u_n\| \rightarrow \infty$  and set  $v_n = \frac{u_n}{\|u_n\|}$ , so that  $\|v_n\| = 1$ . We may assume that  $v_n \rightharpoonup v_0$ , and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Then, from

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_n) \leq \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(v_n) \|u_n\|^{q-2} + C \|u_n\|^{-2},$$

we infer that  $\limsup E(v_n) \leq 0$ . Lemma 2.1(1) yields that  $v_0$  is a constant, and  $v_n \rightarrow v_0$  in  $X$ , which implies  $\|v_0\| = 1$ . On the other hand, since  $u_n \in N_\lambda^-$ , we have  $u_n \in A^+$ , so that  $v_n \in A^+$  and consequently  $v_0 \in A_0^+$ , which is impossible if  $\int_\Omega a < 0$ . Let us assume now  $\int_\Omega b < 0$ . Since

$$E(u_n) > \lambda \frac{p-q}{p-2} B(u_n),$$

from (2.4) we obtain

$$\lambda \frac{(p-q)(q-2)}{2pq} B(u_n) \leq C,$$

so that

$$\lambda \frac{(p-q)(q-2)}{2pq} B(v_n) \leq C \|u_n\|^{-q}.$$

It follows that  $v_0 \in B_0^+$  and we get a contradiction. Hence  $(u_n)$  is bounded. We may then assume that  $u_n \rightharpoonup u_0$  in  $X$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . If  $u_0 \equiv 0$  then we set  $v_n = \frac{u_n}{\|u_n\|}$ . From

$$E(u_n) < \frac{p-q}{2-q} A(u_n),$$

we get

$$E(v_n) < \frac{p-q}{2-q} A(v_n) \|u_n\|^{p-2} \rightarrow 0.$$

So we can assume that  $v_n \rightarrow v_0$  with  $v_0$  constant and non-zero. Moreover, from

$$E(u_n) = \lambda B(u_n) + A(u_n)$$

we deduce that  $B(v_n) \rightarrow 0$ , i.e.  $B(v_0) = 0$ , so  $v_0 \in B_0 \cap A_0^+$ , which contradicts our assumption. Thus  $u_0 \not\equiv 0$ . Note also that  $E(u_0) \leq \frac{p-q}{2-q} A(u_0)$ . We claim that  $u_0 \in A^+$ . Indeed, if  $u_0 \in A_0$  then  $E(u_0) = 0$  i.e.  $u_0$  is a non-zero constant. Hence  $\int_\Omega a = 0$ . By (1.8) we must have  $\int_\Omega b < 0$  and consequently  $u_0 \in B^-$ . But from  $E(u_n) = A(u_n) + \lambda B(u_n)$  we obtain  $E(u_0) \leq \lambda B(u_0)$ , and consequently  $B(u_0) \geq 0$ , which is a contradiction. Therefore  $u_0 \in A^+$ . Furthermore, if  $E(u_0) = 0$  then  $u_0$  is a non-zero constant so  $u_0 \in B^-$ . Summing up, we have either  $u_0 \in A^+ \cap B^-$  or  $u_0 \in A^+ \cap B_0^+ \cap E^+$ . By Proposition 2.3 we infer the

existence of  $t_2 > 0$  such that  $t_2 u_0 \in N_\lambda^-$ . Assume now  $u_n \not\rightarrow u_0$ . Then, since  $u_n \in N_\lambda^-$ , we get

$$I_\lambda(t_2 u_0) < \liminf I_\lambda(t_2 u_n) \leq \liminf I_\lambda(u_n) = \inf_{N_\lambda^-} I_\lambda,$$

which is a contradiction. Therefore  $u_n \rightarrow u_0$ . In particular, we get  $j'_{u_0}(1) = 0$  and  $j''_{u_0}(1) \leq 0$ . Since  $N_\lambda^0$  is empty for  $\lambda \in (0, \lambda_0)$ , we infer that  $u_0 \in N_\lambda^-$  and  $I_\lambda(u_0) = \inf_{N_\lambda^-} I_\lambda$ . We set

$$u_{2,\lambda} = u_0. \quad \square$$

We discuss now the asymptotic profiles of  $u_{1,\lambda}, u_{2,\lambda}$  as  $\lambda \rightarrow 0^+$ .

**Lemma 2.9.** *Assume  $\int_\Omega b < 0$ . Then, for  $0 < \lambda < \lambda_0$ , there holds*

$$I_\lambda(u_{1,\lambda}) < -D_0 \lambda^{\frac{2}{2-q}} - D_1 \lambda^{\frac{p}{2-q}}, \quad (2.5)$$

for some  $D_0 > 0$  and  $D_1 \geq 0$ .

*Proof.* We have  $I_\lambda(u_{1,\lambda}) \leq I_\lambda(u)$  for any  $u \in N_\lambda^+$ . Let us take  $u \in N_\lambda^+$ . Thus  $u \in B^+$  and it follows that  $u$  is non-constant, i.e.  $u \in E^+$ .

- If  $u \in A_0^+$  then

$$I_\lambda(u) \leq \tilde{I}_\lambda(u) := \frac{1}{2}E(u) - \frac{\lambda}{q}B(u).$$

Thus  $I_\lambda(tu) \leq \tilde{I}_\lambda(tu)$  for every  $t > 0$ . Note that  $\tilde{I}_\lambda(tu)$  has a global minimum point  $t_0$  given by

$$t_0 = \left( \frac{\lambda B(u)}{E(u)} \right)^{\frac{1}{2-q}}.$$

and

$$\tilde{I}_\lambda(t_0 u) = -\frac{2-q}{2q} \lambda t_0^q B(u) = -\frac{2-q}{2q} \frac{(\lambda B(u))^{\frac{2}{2-q}}}{E(u)^{\frac{q}{2-q}}} = -D_0 \lambda^{\frac{2}{2-q}},$$

where  $D_0 = \frac{2-q}{2q} \frac{B(u)^{\frac{2}{2-q}}}{E(u)^{\frac{q}{2-q}}}$ . It follows that

$$I_\lambda(u) \leq \tilde{I}_\lambda(t_0 u) = -D_0 \lambda^{\frac{2}{2-q}}$$

with  $D_0 > 0$ .

- If  $u \in A^-$  then we consider  $t_0$  as in the previous item to obtain

$$I_\lambda(u) \leq I_\lambda(t_0 u) = -D_0 \lambda^{\frac{2}{2-q}} - D_1 \lambda^{\frac{p}{2-q}},$$

where  $D_0 = \frac{2-q}{2q} \frac{B(u)^{\frac{2}{2-q}}}{E(u)^{\frac{q}{2-q}}}$  and  $D_1 = \frac{1}{p} \left( \frac{B(u)}{E(u)} \right)^{\frac{p}{2-q}} A(u)$ .

Therefore in both cases we have

$$I_\lambda(u_{1,\lambda}) < -D_0 \lambda^{\frac{2}{2-q}} - D_1 \lambda^{\frac{p}{2-q}}$$

for some  $D_0 > 0$  and  $D_1 \geq 0$ . □

We determine now the asymptotic profile of  $u_{1,\lambda}$  as  $\lambda \rightarrow 0^+$ :

**Proposition 2.10.** *Assume (1.8) and  $\Omega_+^b \neq \emptyset$ . Then  $u_{1,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . More precisely:*

- (1) If  $\int_{\Omega} b < 0$  and  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $\lambda_n^{-\frac{1}{2-q}} u_{1,\lambda_n} \rightarrow w_0$  in  $C^2(\overline{\Omega})$ , where  $w_0$  is a nontrivial non-negative ground state solution of

$$\begin{cases} -\Delta w = b(x)|w|^{q-2}w & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

- (2) If  $\int_{\Omega} a < 0 \leq \int_{\Omega} b$  then  $\lambda^{-\frac{1}{p-q}} u_{1,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . In particular, if  $\int_{\Omega} a < 0 < \int_{\Omega} b$  then  $u_{1,\lambda}$  is positive on  $\overline{\Omega}$  for  $\lambda > 0$  sufficiently small.

*Proof.* First we show that  $u_{1,\lambda}$  remains bounded in  $X$  as  $\lambda \rightarrow 0^+$ . Indeed, assume that  $\lambda_n \rightarrow 0$  and  $\|u_n\| \rightarrow \infty$ , where  $u_n = u_{1,\lambda_n}$ . We set  $v_n = \frac{u_n}{\|u_n\|}$  and assume that for some  $v_0 \in X$  we have  $v_n \rightharpoonup v_0$  in  $X$ , and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Since  $u_n \in N_{\lambda_n}$ , we have

$$E(v_n)\|u_n\|^{2-p} = A(v_n) + \lambda_n B(v_n)\|u_n\|^{q-p}.$$

Passing to the limit we obtain  $A(v_0) = 0$ , i.e.  $v_0 \in A_0$ . From  $u_n \in N_{\lambda_n}^+$  we have

$$E(v_n) < \lambda_n \frac{p-q}{p-2} B(v_n)\|u_n\|^{q-2},$$

so that  $\limsup E(v_n) \leq 0$ . By Lemma 2.1(1) we infer that  $v_n \rightarrow v_0$  in  $X$  and  $v_0$  is a non-zero constant. But  $v_0 \in A_0 \cap B_0^+$ , which contradicts Lemma 2.1(2). Therefore  $(u_n)$  is bounded in  $X$ .

Hence we may assume that  $u_n \rightharpoonup u_0$  in  $X$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . From

$$E(u_n) < \lambda_n \frac{p-q}{p-2} B(u_n). \quad (2.6)$$

we get  $\limsup E(u_n) \leq 0$ . Lemma 2.1(2) provides that  $u_0$  is a constant and  $u_n \rightarrow u_0$  in  $X$ . Since  $u_n \in B^+$ , we have  $u_0 \in B_0^+$ . Finally, from

$$E(u_n) = \lambda_n B(u_n) + A(u_n)$$

we infer that

$$0 = E(u_0) \leq A(u_0),$$

i.e.  $u_0 \in A_0^+$ . Thus  $u_0 \in A_0^+ \cap B_0^+$ , and by Lemma 2.1(2) we deduce that  $u_0 \equiv 0$ . Thus we have proved that  $u_n \rightarrow 0$  in  $X$ . By standard regularity we get  $u_n \rightarrow 0$  in  $C^2(\overline{\Omega})$ .

Next we obtain the precise profile of  $u_n$ . We consider two cases:

- (1) Assume  $\int_{\Omega} b < 0$ . Let  $w_n = \lambda_n^{-\frac{1}{2-q}} u_n$ . We claim that  $(w_n)$  is bounded in  $X$ . Indeed, from (2.6) we have

$$E(w_n) < \frac{p-q}{p-2} B(w_n).$$

Let us assume that  $\|w_n\| \rightarrow \infty$  and set  $\psi_n = \frac{w_n}{\|w_n\|}$ . We may assume that  $\psi_n \rightharpoonup \psi_0$  and  $\psi_n \rightarrow \psi_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows that

$$E(\psi_n) < \frac{p-q}{p-2} B(\psi_n)\|w_n\|^{q-2},$$

so that  $\limsup E(\psi_n) \leq 0$ . By Lemma 2.1(1) we infer that  $\psi_0$  is a constant and  $\psi_n \rightarrow \psi_0$  in  $X$ . On the other hand, from  $u_n \in B^+$  we have  $\psi_n \in B^+$  and consequently  $\psi_0 \in B_0^+$ . From (1.8) we infer that  $\psi_0 \equiv 0$ , which contradicts  $\|\psi_0\| = 1$ . Hence  $(w_n)$  is bounded in  $X$  and we may assume that  $w_n \rightharpoonup w_0$  in  $X$  and  $w_n \rightarrow w_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Note that  $w_n$  satisfies

$$\int_{\Omega} \nabla w_n \nabla w - \lambda_n^{\frac{p-2}{2-q}} \int_{\Omega} a(x) w_n^{p-1} w - \int_{\Omega} b(x) w_n^{q-1} w = 0, \quad \forall w \in X. \quad (2.7)$$

Taking  $w = w_n - w_0$  we deduce that  $w_n \rightarrow w_0$  in  $X$  and consequently in  $C^2(\overline{\Omega})$ . Moreover,  $w_0$  is a solution of (1.10). We claim that  $w_0 \not\equiv 0$ . Indeed, by Lemma 2.9 we have

$$I_{\lambda_n}(u_n) < -D_0 \lambda_n^{\frac{2}{2-q}} - D_1 \lambda_n^{\frac{p}{2-q}},$$

with  $D_0 > 0$  and  $D_1 \geq 0$ . Hence

$$\frac{\lambda_n^{\frac{2}{2-q}}}{2} E(w_n) - \frac{\lambda_n^{\frac{p}{2-q}}}{p} A(w_n) - \frac{\lambda_n^{\frac{2}{2-q}}}{q} B(w_n) < -D_0 \lambda_n^{\frac{2}{2-q}} - D_1 \lambda_n^{\frac{p}{2-q}},$$

so that

$$\frac{1}{2} E(w_n) - \frac{\lambda_n^{\frac{p-2}{2-q}}}{p} A(w_n) - B(w_n) < -D_0 - D_1 \lambda_n^{\frac{p-2}{2-q}}.$$

We obtain then

$$\frac{1}{2} E(w_0) - B(w_0) \leq -D_0,$$

and consequently  $w_0 \not\equiv 0$ .

It remains to prove that  $w_0$  is a ground state solution of (1.10), i.e.

$$I_b(w_0) = \min_{N_b} I_b,$$

where

$$I_b(u) = \frac{1}{2} E(u) - \frac{1}{q} B(u)$$

for  $u \in X$  and

$$N_b = \{u \in X \setminus \{0\}; \langle I'_b(u), u \rangle = 0\} = \{u \in X \setminus \{0\}; E(u) = B(u)\}$$

is the Nehari manifold associated to  $I_b$ . Since  $\int_{\Omega} b < 0$  it is easily seen that there exists  $w_b \neq 0$  such that  $I_b(w_b) = \min_{N_b} I_b$ . Note that since  $w_0$  is a nontrivial solution of (1.10) we have  $w_0 \in N_b$  and consequently  $I_b(w_b) \leq I_b(w_0)$ . We prove now the reverse inequality. Since  $w_b$  is non-constant, we have  $w_b \in B^+ \cap E^+$ . We set  $u_b = \lambda^{\frac{1}{2-q}} w_b$ . Let  $\lambda_n \rightarrow 0^+$ . Since  $u_b \in B^+ \cap E^+$  for every  $n$  there exists  $t_n > 0$  such that  $t_n u_b \in N_{\lambda_n}^+$ . Hence

$$t_n^2 E(u_b) < \lambda_n \frac{p-q}{p-2} t_n^q B(u_b),$$

i.e.

$$t_n^{2-q} < \frac{p-q}{p-2} \frac{B(w_b)}{E(w_b)} = \frac{p-q}{p-2}.$$

We may then assume that  $t_n \rightarrow t_0$ . We claim that  $t_0 = 1$ . Indeed, note that from  $t_n u_b \in N_{\lambda_n}^+$  we infer that

$$t_n^2 E(u_b) = \lambda_n t_n^q B(u_b) + t_n^p A(u_b)$$

so

$$t_n^{2-q} E(w_b) = B(w_b) + t_n^{p-q} \lambda_n^{\frac{p-2}{2-q}} A(w_b).$$

From  $E(w_b) = B(w_b)$  we infer that  $t_0 = 1$ , as claimed. Now, from

$$I_{\lambda_n}(u_{1,\lambda_n}) \leq I_{\lambda_n}(t_n u_b)$$

it follows that

$$I_{\lambda_n}(u_{1,\lambda_n}) \leq \left(\frac{1}{2} - \frac{1}{q}\right) t_n^2 E(u_b) - \left(\frac{1}{p} - \frac{1}{q}\right) t_n^p A(u_b).$$

Hence

$$\frac{\lambda_n^{\frac{2}{2-q}}}{2} E(w_n) - \frac{\lambda_n^{\frac{p}{2-q}}}{p} A(w_n) - \frac{\lambda_n^{\frac{2}{2-q}}}{q} B(w_n) \leq \frac{q-2}{2q} t_n^2 \lambda_n^{\frac{2}{2-q}} E(w_b) - \frac{q-p}{pq} \lambda_n^{\frac{p}{2-q}} t_n^p A(w_b),$$



i.e.

$$\frac{1}{2}E(w_n) - \frac{\lambda_n^{\frac{p-2}{2-q}}}{p}A(w_n) - \frac{1}{q}B(w_n) \leq \frac{q-2}{2q}t_n^2E(w_b) - \frac{q-p}{pq}\lambda_n^{\frac{p-2}{2-q}}t_n^pA(w_b).$$

Since  $w_n \rightarrow w_0$  in  $X$  we obtain

$$I_b(w_0) \leq \left(\frac{1}{2} - \frac{1}{q}\right)E(w_b) = I_b(w_b).$$

Therefore  $I_b(w_0) = I_b(w_b)$ , as claimed.

- (2) Assume now  $\int_{\Omega} a < 0 \leq \int_{\Omega} b$  and set  $w_n = \lambda_n^{-\frac{1}{p-q}}u_n$ . We claim that  $(w_n)$  is bounded in  $X$ . Indeed, since  $u_n \in N_{\lambda_n}^+$ , we have

$$E(w_n) < \frac{p-q}{p-2}\lambda_n^{\frac{p-2}{p-q}}B(w_n).$$

Let us assume that  $\|w_n\| \rightarrow \infty$  and set  $\psi_n = \frac{w_n}{\|w_n\|}$ . We may assume that  $\psi_n \rightharpoonup \psi_0$  and  $\psi_n \rightarrow \psi_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows that

$$E(\psi_n) < \frac{p-q}{p-2}\lambda_n^{\frac{p-2}{p-q}}B(\psi_n)\|w_n\|^{q-2},$$

so that  $\limsup E(\psi_n) \leq 0$ . By Lemma 2.1(1) we infer that  $\psi_0$  is a constant and  $\psi_n \rightarrow \psi_0$  in  $X$ . On the other hand, from  $u_n \in N_{\lambda_n}$  it follows that

$$0 \leq A(u_n) + \lambda_n B(u_n),$$

so that

$$-B(\psi_n)\|w_n\|^{q-p} \leq A(\psi_n).$$

Taking the limit we get  $0 \leq A(\psi_0)$ , which contradicts Lemma 2.1(2). Hence  $(w_n)$  is bounded in  $X$  and we may assume that  $w_n \rightharpoonup w_0$  in  $X$  and  $w_n \rightarrow w_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows that  $\limsup E(w_n) \leq 0$ , and by Lemma 2.1(1) we get that  $w_0$  is a constant and  $w_n \rightarrow w_0$  in  $X$ . So  $w_n \rightarrow w_0$  in  $C^2(\overline{\Omega})$ .

It remains to show that  $w_0 = c^*$ . We note that  $w_n$  satisfies

$$\int_{\Omega} \nabla w_n \nabla w - \lambda_n^{\frac{p-2}{p-q}} \int_{\Omega} a w_n^{p-1} w - \lambda_n^{\frac{p-2}{p-q}} \int_{\Omega} b w_n^{q-1} w = 0, \quad \forall w \in X, \quad (2.8)$$

since  $u_n$  is a solution of  $(P_{\lambda_n})$ . We infer that

$$\int_{\Omega} a w_n^{p-1} + \int_{\Omega} b w_n^{q-1} = 0.$$

Passing to the limit, we see that either  $w_0 = 0$  or  $w_0 = c^*$ . However, taking now  $w = (w_n + \varepsilon)^{1-q}$  with  $\varepsilon > 0$  in (2.8), we obtain

$$0 > (1-q) \int_{\Omega} |\nabla w_n|^2 (w_n + \varepsilon)^{-q} = \lambda_n^{\frac{p-2}{p-q}} \left( \int_{\Omega} a \frac{w_n^{p-1}}{(w_n + \varepsilon)^{q-1}} + \int_{\Omega} b \left( \frac{w_n}{w_n + \varepsilon} \right)^{q-1} \right).$$

so that

$$\int_{\Omega} b \left( \frac{w_n}{w_n + \varepsilon} \right)^{q-1} < - \int_{\Omega} a \frac{w_n^{p-1}}{(w_n + \varepsilon)^{q-1}}.$$

Letting  $\varepsilon \rightarrow 0$  and using the Lebesgue dominated convergence theorem, we get

$$\int_{\text{supp } w_n} b \leq - \int_{\Omega} a w_n^{p-q}.$$

In particular, since  $b \leq 0$  on  $\Omega \setminus \text{supp } w_n$ , we have

$$\int_{\Omega} b \leq - \int_{\Omega} a w_n^{p-q}.$$

Letting now  $n \rightarrow \infty$ , we obtain

$$0 \leq \int_{\Omega} b \leq - \int_{\Omega} a w_0^{p-q}.$$

Now, if  $\int_{\Omega} b = 0$  then  $c^* = 0$ , so  $w_0 = 0$ . On the other hand, if  $\int_{\Omega} b > 0$  then we must have  $w_0 \neq 0$ , i.e.  $w_0 = c^*$ . In both cases we obtain  $\lambda_n^{-\frac{1}{p-q}} u_n \rightarrow c^*$  in  $X$ . By elliptic regularity we deduce that  $\lambda^{-\frac{1}{p-q}} u_{1,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$ . Additionally if  $\int_{\Omega} a < 0 < \int_{\Omega} b$  then  $c^* > 0$ , so that, by continuity,  $u_{1,\lambda} > 0$  on  $\overline{\Omega}$  for  $\lambda > 0$  sufficiently small.

□

We consider now the asymptotic behavior of  $u_{2,\lambda}$  as  $\lambda \rightarrow 0^+$ :

**Lemma 2.11.** *Assume (1.8) and  $\Omega_+^a \neq \emptyset$ . Then there exists a constant  $C > 0$  such that  $\|u_{2,\lambda}\| \leq C$  as  $\lambda \rightarrow 0^+$ .*

*Proof.* First we show that there exists a constant  $C_1 > 0$  such that  $I_{\lambda}(u_{2,\lambda}) \leq C_1$  for every  $\lambda \in (0, \lambda_0)$ . To this end, we consider the following eigenvalue problem with the Dirichlet boundary condition.

$$\begin{cases} -\Delta \varphi = \lambda a(x) \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.9)$$

We denote by  $\lambda_D = \lambda_D(\Omega)$  the positive principal eigenvalue of (2.9) and by  $\varphi_D = \varphi_D(\Omega)$  a positive eigenfunction associated to  $\lambda_D$ . Taking  $\varphi_D^{p-1}$  as test function, we see that  $\int_{\Omega} a \varphi_D^p = \int_{\Omega} |\nabla \varphi_D|^2 > 0$ , i.e.  $\varphi_D \in E^+ \cap A^+$ . By Proposition 2.3 there exists  $t_2(\lambda)$  such that  $t_2(\lambda) \varphi_D \in N_{\lambda}^-$ . Note that

$$0 < j_{\varphi_D}(t_2(\lambda)) = \frac{t_2(\lambda)^2}{2} E(\varphi_D) - \frac{t_2(\lambda)^p}{p} A(\varphi_D) - \lambda \frac{t_2(\lambda)^q}{q} B(\varphi_D).$$

Thus  $t_2(\lambda)$  stays bounded as  $\lambda \rightarrow 0^+$ . Consequently, there holds

$$I_{\lambda}(t_2(\lambda) \varphi_D) = \frac{q-2}{2q} t_2(\lambda)^2 E(\varphi_D) + \frac{p-q}{pq} t_2(\lambda)^p A(\varphi_D) \leq \frac{p-q}{pq} t_2(\lambda)^p A(\varphi_D) \leq C,$$

for some constant  $C > 0$ .

We assume now that  $\lambda_n \rightarrow 0^+$  and  $\|u_n\| \rightarrow \infty$ , where  $u_n = u_{2,\lambda_n}$ . We set  $v_n = \frac{u_n}{\|u_n\|}$  and assume that  $v_n \rightarrow v_0$  in  $X$  and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Since  $I_{\lambda_n}(u_n) = \min_{N_{\lambda_n}^-} I_{\lambda_n}$ , we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \left(\frac{1}{q} - \frac{1}{p}\right) \lambda_n B(u_n) = I_{\lambda_n}(u_n) \leq C.$$

Hence

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_n) \leq \left(\frac{1}{q} - \frac{1}{p}\right) \lambda_n B(v_n) \|u_n\|^{q-2} + C_1 \|u_n\|^{-2}.$$

Letting  $\lambda_n \rightarrow 0^+$  we obtain  $\limsup_{\lambda} E(v_n) \leq 0$ , and by Lemma 2.1 we infer that  $v_0$  is a constant and  $v_n \rightarrow v_0$  in  $X$ . In particular,  $\|v_0\| = 1$ . Moreover, from

$$\int_{\Omega} (\nabla u_n \nabla \phi - \lambda_n b u_n^{q-1} \phi - a u_n^{p-1} \phi) = 0 \quad \forall \phi \in X,$$

we get

$$\int_{\Omega} a v_0^{p-1} \phi = \lim \int_{\Omega} a v_n^{p-1} \phi = 0 \quad \forall \phi \in X,$$

which provides  $av_0^{p-1} \equiv 0$ , so that  $v_0 = 0$ , and we get a contradiction. Therefore  $(u_{2,\lambda})$  stays bounded in  $X$  as  $\lambda \rightarrow 0^+$ .  $\square$

**Proposition 2.12.** *Assume (1.8) and  $\Omega_+^a \neq \emptyset$ .*

- (1) *If  $\int_\Omega a \geq 0 > \int_\Omega b$  then  $u_{2,\lambda} \rightarrow 0$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . If, in addition,  $\int_\Omega a > 0$  then  $\lambda^{-\frac{1}{p-q}} u_{2,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . In this case  $u_{2,\lambda}$  is a unstable positive solution of  $(P_\lambda)$  for  $\lambda > 0$  sufficiently small.*
- (2) *If  $\int_\Omega a < 0$  and  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $u_{2,\lambda_n} \rightarrow u_{2,0}$  in  $C^2(\overline{\Omega})$ , where  $u_{2,0}$  is a positive ground state solution of (1.11). In this case  $u_{2,\lambda}$  is a unstable positive solution of  $(P_\lambda)$  for  $\lambda > 0$  sufficiently small.*

*Proof.* Let  $\lambda_n \rightarrow 0^+$ . By Lemma 2.11, up to a subsequence, we have  $u_n = u_{2,\lambda_n} \rightharpoonup u_0$  in  $X$  and  $u_n \rightarrow u_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Since  $u_n$  is a solution of  $(P_{\lambda_n})$  it follows that  $u_n \rightarrow u_0$  in  $C^2(\overline{\Omega})$  and  $u_0$  is a non-negative solution of (1.11). This problem has a nontrivial non-negative solution if and only if  $\int_\Omega a < 0$ . Hence  $u_0 \equiv 0$  if  $\int_\Omega a \geq 0$ .

- (1) Let us now assume that  $\int_\Omega a > 0 > \int_\Omega b$ . We set  $w_n = \lambda_n^{-\frac{1}{p-q}} u_n$ . Then  $w_n$  is a non-negative solution of

$$\begin{cases} -\Delta w = \lambda_n^{\frac{p-2}{p-q}} a(x) w^{p-1} + \lambda_n^{\frac{p-2}{p-q}} b(x) w^{q-1} & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.10)$$

for  $\lambda = \lambda_n$ . We claim that  $(w_n)$  is bounded in  $X$ . Indeed, assume that  $\|w_n\| \rightarrow \infty$  and  $\psi_n = \frac{w_n}{\|w_n\|} \rightharpoonup \psi_0$  in  $X$  with  $\psi_n \rightarrow \psi_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Let  $c_\lambda = \left( \frac{-\lambda \int_\Omega b}{\int_\Omega a} \right)^{\frac{1}{p-q}}$ . We use now the fact that  $c_\lambda \in N_\lambda^-$  for any  $\lambda > 0$ . Hence

$$I_{\lambda_n}(u_n) \leq I_{\lambda_n}(c_{\lambda_n}) = D \lambda_n^{\frac{p}{p-q}},$$

where  $D = \frac{p-q}{pq} \frac{(-\int_\Omega b)^{\frac{p}{p-q}}}{(\int_\Omega a)^{\frac{q}{p-q}}}$ . Thus

$$\frac{p-2}{2p} \lambda_n^{\frac{2}{p-q}} E(w_n) - \frac{p-q}{pq} \lambda_n^{\frac{p}{p-q}} B(w_n) \leq D \lambda_n^{\frac{p}{p-q}},$$

so that

$$\frac{p-2}{2p} E(w_n) - \frac{p-q}{pq} \lambda_n^{\frac{p-2}{p-q}} B(w_n) \leq D \lambda_n^{\frac{p-2}{p-q}}.$$

Dividing the latter inequality by  $\|w_n\|^2$  we get  $E(\psi_n) \rightarrow 0$ , and consequently  $\psi_n \rightarrow \psi_0$  in  $X$  and  $\psi_0$  is a constant. Furthermore, integrating (2.10) we obtain

$$\int_\Omega a w_n^{p-1} + \int_\Omega b w_n^{q-1} = 0, \quad (2.11)$$

so that  $\int_\Omega a \psi_n^{p-1} \rightarrow 0$ , i.e.  $\int_\Omega a \psi_0^{p-1} = 0$ , and consequently  $\int_\Omega a = 0$ , which is a contradiction. Therefore  $(w_n)$  is bounded in  $X$ . We may assume then that  $w_n \rightharpoonup w_0$  in  $X$  and  $w_n \rightarrow w_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . It follows that

$$\int_\Omega \nabla w_0 \nabla \phi = 0, \quad \forall \phi \in X.$$

Hence  $w_0$  is a constant and  $w_n \rightarrow w_0$  in  $X$ , and consequently in  $C^2(\overline{\Omega})$ . It remains to show that  $w_0 \neq 0$ . If  $w_0 = 0$  then we set again  $\psi_n = \frac{w_n}{\|w_n\|}$ . From

$$E(w_n) < \frac{p-2}{p-q} \lambda_n^{\frac{p-2}{p-q}} A(w_n),$$

we infer that  $E(\psi_n) \rightarrow 0$ , so that  $\psi_n \rightarrow \psi_0$  in  $X$  and  $\psi_0$  is a constant. Moreover, from

$$0 \leq A(w_n) + B(w_n)$$

we have

$$-\|w_n\|^{p-q} A(\psi_n) \leq B(\psi_n),$$

so that  $B(\psi_0) \geq 0$ . By Lemma 2.1(2) we get a contradiction. Therefore we have proved that  $w_0$  is a non-zero constant. Finally, from (2.11) we obtain

$$w_0^{p-1} \int_{\Omega} a = -w_0^{q-1} \int_{\Omega} b,$$

i.e.  $w_0 = c^*$ . In particular, we infer that  $u_{2,\lambda}$  is positive for  $\lambda > 0$  sufficiently small. Finally, from Remark 2.2 we infer that  $u_{2,\lambda}$  is unstable whenever it is positive.

- (2) Let us assume now  $\int_{\Omega} a < 0$  and show that  $u_{2,0}$  is a positive ground state solution of (1.11), i.e.

$$I_a(u_{2,0}) = \min_{N_a} I_a,$$

where

$$I_a(u) = \frac{1}{2}E(u) - \frac{1}{p}A(u)$$

for  $u \in X$  and

$$N_a = \{u \in X \setminus \{0\}; \langle I'_a(u), u \rangle = 0\} = \{u \in X \setminus \{0\}; E(u) = A(u)\}$$

is the Nehari manifold associated to  $I_a$ . Since  $\int_{\Omega} a < 0$  it is easily seen that there exists  $u_a \neq 0$  such that  $I_a(u_a) = \min_{N_a} I_a$ . Note that since  $u_{2,0}$  is a nontrivial solution of (1.11) we have  $u_{2,0} \in N_a$  and consequently  $I_a(u_a) \leq I_a(u_{2,0})$ . We prove now the reverse inequality. Since  $u_a$  is non-constant, we have  $u_a \in A^+ \cap E^+$ . Thus for any  $\lambda < \lambda_0$  there exists  $t_\lambda > 0$  such that  $t_\lambda u_0 \in N_\lambda^-$ . Thus

$$t_\lambda^2 \frac{E(u_a)}{2} - \lambda t_\lambda^q \frac{B(u_a)}{q} - t_\lambda^p \frac{A(u_a)}{p} = I_\lambda(t_\lambda u_a) > 0,$$

which implies that  $t_\lambda$  remains bounded as  $\lambda \rightarrow 0$ . We may then assume that  $t_\lambda \rightarrow t_0$  as  $\lambda \rightarrow 0$ . We claim that  $t_0 = 1$ . Indeed, first note that from  $t_\lambda u_a \in N_\lambda^-$  we have

$$t_\lambda^2 E(u_a) < \frac{p-q}{2-q} t_\lambda^p A(u_a),$$

and consequently  $t_\lambda^{p-2} > \frac{2-q}{p-q} \frac{E(u_a)}{A(u_a)}$ . Hence  $t_0 > 0$ . In addition, from  $t_\lambda u_a \in N_\lambda$  we have

$$t_\lambda^2 E(u_a) = \lambda t_\lambda^q B(u_a) + t_\lambda^p A(u_a)$$

so

$$t_\lambda^{2-q} E(u_a) = t_\lambda^{p-q} A(u_a) + o(1)$$

as  $\lambda \rightarrow 0$ . Since  $E(u_a) = A(u_a)$  we infer that  $t_0 = 1$ , as claimed. Now, from

$$I_\lambda(u_{2,\lambda}) \leq I_\lambda(t_\lambda u_a)$$

it follows that

$$\frac{1}{2}E(u_{2,\lambda}) - \frac{\lambda}{q}B(u_{2,\lambda}) - \frac{1}{p}A(u_{2,\lambda}) \leq \left(\frac{1}{2} - \frac{1}{p}\right) t_\lambda^2 E(u_a) - \left(\frac{1}{q} - \frac{1}{p}\right) \lambda t_\lambda^q B(u_a).$$

Letting  $\lambda \rightarrow 0$  and using that  $u_{2,\lambda} \rightarrow u_{2,0}$  in  $X$  we obtain

$$I_a(u_{2,0}) \leq \left(\frac{1}{2} - \frac{1}{p}\right) E(u_a) = I_a(u_a).$$

Therefore  $I_a(u_{2,0}) = I_a(u_a)$ , as claimed.

Finally, let us show that  $u_{2,\lambda}$  is positive on  $\overline{\Omega}$  for  $\lambda > 0$  sufficiently small. Indeed, assume by contradiction that for every  $\lambda > 0$  there exists  $0 < \mu < \lambda$  such that  $u_{2,\mu}$  is non-negative but vanishes somewhere on  $\overline{\Omega}$ . Then we obtain a sequence  $\mu_n \rightarrow 0^+$  such that  $u_n = u_{2,\mu_n}$  are non-negative solutions vanishing somewhere on  $\overline{\Omega}$ . But up to a subsequence,  $(u_n)$  converges in  $C^2(\overline{\Omega})$  to a positive function, which is a contradiction. The proof is now complete.  $\square$

**Remark 2.13.**

- (1) If  $\Omega_+^a \neq \emptyset$  and  $\int_{\Omega} a < 0 < \int_{\Omega} b$  then Propositions 2.10 and 2.12 provide us with some  $\lambda^* > 0$  such that  $u_{2,\lambda} > u_{1,\lambda}$  for  $0 < \lambda < \lambda^*$ . Indeed, assume on the contrary that there are two sequences  $\lambda_n \rightarrow 0^+$  and  $x_n \in \overline{\Omega}$  such that  $u_{2,\lambda_n}(x_n) \leq u_{1,\lambda_n}(x_n)$  and  $u_{2,\lambda_n} \rightarrow u_{2,0}$  in  $C^2(\overline{\Omega})$ . Let  $\epsilon = \frac{\min_{\overline{\Omega}} u_{2,0}}{2} > 0$ . Since  $u_{1,\lambda_n} \rightarrow 0$  in  $C^2(\overline{\Omega})$ , there exists  $n_0 \in \mathbb{N}$  such that  $\max_{\overline{\Omega}} u_{1,\lambda_n} < \epsilon$  for  $n \geq n_0$ . It follows that  $u_{1,\lambda_n}(x_n) < \epsilon$  for  $n \geq n_0$ , and thus that  $u_{2,\lambda_n}(x_n) < \epsilon$  for such  $n$ . However, since  $u_{2,\lambda_n} \rightarrow u_{2,0}$  in  $C^2(\overline{\Omega})$ , there exists  $n_1 \in \mathbb{N}$  such that  $\min_{\overline{\Omega}} u_{2,\lambda_n} > \epsilon$  for  $n \geq n_1$ , which is a contradiction. The same result holds if  $\Omega_+^b \neq \emptyset$  and  $\int_{\Omega} a > 0 > \int_{\Omega} b$ . Indeed, one can apply a similar argument to  $w_{1,\lambda} = \lambda^{-\frac{1}{p-q}} u_{1,\lambda}$  and  $w_{2,\lambda} = \lambda^{-\frac{1}{p-q}} u_{2,\lambda}$ . We use now the fact that  $w_{2,\lambda} \rightarrow c^*$  in  $C^2(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$  and if  $\lambda_n \rightarrow 0^+$  then, up to a subsequence,  $w_{1,\lambda_n} \rightarrow 0$  in  $C^2(\overline{\Omega})$ .
- (2) One can show that ground state solutions of (1.11) converge to 0 in  $C^2(\overline{\Omega})$  as  $\int_{\Omega} a \nearrow 0$ . More precisely, let  $a_n = a^+ - \delta_n a^-$  where  $\delta_n$  is a sequence such that  $\delta_n \searrow \delta_0 = \frac{\int_{\Omega} a^+}{\int_{\Omega} a^-}$ . Then  $a_n \nearrow a_0 = a^+ - \delta_0 a^-$  and  $\int_{\Omega} a_0 = 0$ . We denote by  $u_n$  a ground state solution of (1.11) with  $a = a_n$ . First we show that  $(u_n)$  is bounded in  $X$ . If not, we set  $v_n = \frac{u_n}{\|u_n\|}$  and assume that  $v_n \rightarrow v_0$  in  $X$ . We set  $A_n(u) = \int_{\Omega} a_n |u|^p$  and  $I_n(u) = \frac{1}{2} E(u) - \frac{1}{p} A_n(u)$  for  $u \in X$ . In addition, we denote by  $N_n$  the Nehari manifold associated to  $I_n$ . Let  $\phi \in X$  be such that  $a^- \phi \equiv 0$  and  $a^+ \phi \not\equiv 0$ . Then

$$I_n(t\phi) = \frac{t^2}{2} E(\phi) - \frac{t^p}{p} \int_{\Omega} a^+ |\phi|^p = C(t).$$

Since  $\phi$  is non-constant there exists a unique  $t_0 > 0$  such that  $t_0 \phi \in N_n$  for every  $n$ . It follows that

$$\frac{p-2}{2p} E(u_n) = I_n(u_n) \leq I_n(t_0 \phi) = C(t_0).$$

Consequently we have  $E(v_n) \rightarrow 0$ , so that  $v_n \rightarrow v_0$  and  $v_0$  is a constant. Moreover, since

$$\int_{\Omega} \nabla u_n \nabla w = \int_{\Omega} a_n u_n^{p-1} w \quad \forall w \in X,$$

we deduce that

$$\int_{\Omega} a_n v_n^{p-1} w \rightarrow 0 \quad \forall w \in X$$

Thus

$$\int_{\Omega} a_0 v_0^{p-1} w = 0 \quad \forall w \in X$$

and consequently  $a_0 v_0^{p-1} \equiv 0$ . Hence  $v_0 = 0$ , which contradicts  $\|v_n\| = 1$  for every  $n$ . Therefore  $(u_n)$  is bounded and, up to a subsequence, we have  $u_n \rightarrow u_0$  in  $X$ .

Moreover  $u_0$  is a solution of (1.11) with  $a = a_0$ . Finally, since  $\int_{\Omega} a_0 = 0$  we infer that  $u_0 \equiv 0$ , i.e.  $u_n \rightarrow 0$  in  $X$ , and consequently in  $C^2(\overline{\Omega})$ .

### 3. SOME RESULTS VIA SUB-SUPERSOLUTIONS

We use now the asymptotic profile of  $u_{1,\lambda}$  as  $\lambda \rightarrow 0$  to show that for  $\lambda > 0$  sufficiently small a solution of  $(P_{\lambda})$  can be obtained by the sub-supersolutions method. In particular, the assumption  $p < 2^*$  can be dropped.

**Proposition 3.1.** *Assume that  $\Omega_+^b \neq \emptyset$  and  $\int_{\Omega} b < 0$ . Then there exists  $\Lambda_0 > 0$  such that  $(P_{\lambda})$  has a nontrivial non-negative solution  $U_{\lambda}$  for  $0 < \lambda < \Lambda_0$ . Moreover  $U_{\lambda} \rightarrow 0$  in  $X$  as  $\lambda \rightarrow 0^+$ .*

*Proof.* First we obtain a supersolution of  $(P_{\lambda})$ . To this end, we consider the problem

$$\begin{cases} -\Delta w = (b(x) + \delta)|w|^{q-2}w & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

If  $\delta > 0$  is such that  $\int_{\Omega} (b + \delta) < 0$  then this problem has a nontrivial non-negative solution  $w_{\delta}$ . We set  $\overline{u} = \lambda^{\frac{1}{2-q}} w_{\delta}$ . Then  $\overline{u}$  is a weak supersolution of  $(P_{\lambda})$  if

$$\lambda^{\frac{1}{2-q}} \int_{\Omega} (b(x) + \delta) w_{\delta}(x)^{q-1} v \geq \lambda^{\frac{p-1}{2-q}} \int_{\Omega} a(x) w_{\delta}(x)^{p-1} v + \lambda^{\frac{1}{2-q}} \int_{\Omega} b(x) w_{\delta}(x)^{q-1} v$$

for every non-negative  $v \in X$ . It suffices then to have

$$\lambda^{\frac{1}{2-q}} (b(x) + \delta) w_{\delta}(x)^{q-1} \geq \lambda^{\frac{p-1}{2-q}} a(x) w_{\delta}(x)^{p-1} + \lambda^{\frac{1}{2-q}} b(x) w_{\delta}(x)^{q-1}$$

for *a.e.*  $x \in \Omega$ . If  $w_{\delta}(x) = 0$  or  $a(x) \leq 0$  then the latter inequality is clearly satisfied. Now, if  $w_{\delta}(x) > 0$  and  $a(x) > 0$  then it is equivalent to

$$\delta \geq \lambda^{\frac{p-2}{2-q}} a(x) w_{\delta}^{p-q},$$

which is satisfied if

$$\lambda \leq \Lambda_0 := (\delta \|a^+\|_{\infty}^{-1} \|w_{\delta}\|_{\infty}^{q-p})^{\frac{2-q}{p-2}}.$$

On the other hand, since  $\Omega_+^b \neq \emptyset$  there exist a subdomain  $\Omega' \subset \Omega$  and  $\delta' > 0$  such that  $b \geq \delta'$  in  $\Omega'$ . Let  $\phi'_1$  be a positive eigenfunction associated to  $\lambda'_1$ , the first eigenvalue of  $-\Delta u = \lambda u$  in  $\Omega'$ ,  $u = 0$  in  $\partial\Omega'$ . We extend  $\phi'_1$  by zero to  $\Omega \setminus \Omega'$  and set  $\underline{u} = \varepsilon \phi'_1$ , where  $\varepsilon > 0$ . Then we have, for a non-negative  $v \in X$ ,

$$\int_{\Omega} \nabla \underline{u} \nabla v = \varepsilon \int_{\Omega'} \nabla \phi'_1 \nabla v = \varepsilon \left( \int_{\partial\Omega'} \frac{\partial \phi'_1}{\partial \mathbf{n}} v + \lambda'_1 \int_{\Omega'} \phi'_1 v \right) \leq \varepsilon \lambda'_1 \int_{\Omega'} \phi'_1 v,$$

since  $\frac{\partial \phi'_1}{\partial \mathbf{n}} < 0$  on  $\partial\Omega'$ . Hence  $\underline{u}$  is a weak subsolution of  $(P_{\lambda})$  if, for *a.e.*  $x \in \Omega'$ , we have

$$\varepsilon \lambda'_1 \phi'_1 \leq a(\varepsilon \phi'_1)^{p-1} + \lambda b(\varepsilon \phi'_1)^{q-1},$$

i.e.

$$\lambda'_1 (\varepsilon \phi'_1)^{2-q} \leq a(\varepsilon \phi'_1)^{p-q} + \lambda b.$$

This inequality is clearly satisfied for  $\varepsilon > 0$  sufficiently small since  $b \geq \delta' > 0$  in  $\Omega'$ . Finally, taking  $\varepsilon > 0$  smaller if necessary, we have  $\varepsilon \phi'_1 \leq \overline{u}$  in  $\Omega$ . By [16, Theorem 2] we deduce that  $(P_{\lambda})$  has a solution  $U_{\lambda}$  which satisfies  $\varepsilon \phi'_1 \leq U_{\lambda} \leq \lambda^{\frac{1}{2-q}} w_{\delta}$  in  $\Omega$  for  $\lambda < \Lambda_0$ . In particular, we have  $U_{\lambda} \rightarrow 0$  in  $C(\overline{\Omega})$ , and consequently in  $X$ , as  $\lambda \rightarrow 0^+$ .  $\square$

We prove now that bifurcation of nontrivial non-negative solutions from zero can not occur at any  $\lambda > 0$ .

**Lemma 3.2.** *Assume that  $\Omega_+^b$  is a subdomain of  $\Omega$ . Let  $\bar{\lambda} > 0$  and  $D$  be a subdomain such that  $\bar{D} \subset \Omega_+^b$ . Then there exists  $C_{\bar{\lambda}} > 0$  such that  $u \geq C_{\bar{\lambda}}$  in  $\bar{D}$  for every  $u \in B^+$  which is a non-negative solution of  $(P_{\lambda})$  for  $\lambda \geq \bar{\lambda}$ .*

*Proof.* We use a variant of a comparison principle for concave problems due to Ambrosetti-Brezis-Cerami [3, Lemma 3.3]. Let  $u \in B^+$  be a nontrivial non-negative solution of  $(P_{\lambda})$  for  $\lambda \geq \bar{\lambda}$ . First we claim that  $u > 0$  in  $\Omega_+^b$ . Indeed, since  $u \in B^+$ , we deduce that  $u$  is positive somewhere in  $\Omega_+^b$ . It follows that there exists a constant  $M > 0$  large such that  $(-\Delta + M)u \geq 0$  and  $(-\Delta + M)u \not\equiv 0$  in  $\Omega_+^b$ . The strong maximum principle provides then the desired conclusion.

We apply now [3, Lemma 3.3] to the following concave problem

$$\begin{cases} -\Delta v = -a_0 v^{p-1} + \lambda b_0 v^{q-1} & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \quad (3.1)$$

where  $a_0 = \sup_D a^-$  and  $b_0 = \inf_D b$ . It is clear that  $u$  is a supersolution of (3.1). Next we construct a subsolution of (3.1). To this end, we use a positive eigenfunction  $\phi_1$  associated to the first eigenvalue  $\lambda_1 > 0$  of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta \phi = \lambda \phi & \text{in } D, \\ \phi = 0 & \text{on } \partial D. \end{cases}$$

We normalize  $\phi_1$  by  $\|\phi_1\|_{C(\bar{D})} = 1$ . Then

$$\begin{aligned} -\Delta(\delta \phi_1) - \{-a_0(\delta \phi_1)^{p-1} + \lambda b_0(\delta \phi_1)^{q-1}\} &\leq (\delta \phi_1)^{q-1} \{\lambda_1 \delta^{2-q} + a_0 \delta^{p-q} - \bar{\lambda} b_0\} \\ &\leq (\delta \phi_1)^{q-1} \{2\lambda_1 \delta^{2-q} - \bar{\lambda} b_0\} \end{aligned}$$

if  $x \in D$ ,  $\lambda \geq \bar{\lambda}$  and  $0 < \delta \leq \bar{\delta}$  for some  $\bar{\delta}$  sufficiently small. Thus  $c_{\bar{\lambda}} \phi_1$  is a subsolution of (3.1) for  $\lambda \geq \bar{\lambda}$  if we set

$$c_{\bar{\lambda}} = \min \left\{ \left( \frac{\bar{\lambda} b_0}{2\lambda_1} \right)^{\frac{1}{2-q}}, \bar{\delta} \right\} > 0.$$

The comparison principle ensures then that  $c_{\bar{\lambda}} \phi_1 \leq u$  in  $\bar{D}$ , from which the desired conclusion follows.  $\square$

**Proposition 3.3.** *Under the assumptions of Lemma 3.2, bifurcation from zero never occurs for  $(P_{\lambda})$  at any  $\lambda > 0$ . More precisely, it never occurs that there exist  $\lambda_n$  and nontrivial non-negative solutions  $u_{\lambda_n}$  of  $(P_{\lambda_n})$  such that  $\lambda_n \rightarrow \lambda^* > 0$  and  $u_n \rightarrow 0$  in  $C(\bar{\Omega})$ .*

*Proof.* Assume by contradiction that  $\lambda_n \rightarrow \lambda^* > 0$  and  $u_n$  is a nontrivial non-negative solution of  $(P_{\lambda_n})$  with  $u_n \rightarrow 0$  in  $C(\bar{\Omega})$ . By Lemma 3.2 we must have  $u_n \in B_0^-$  for  $n$  large enough. Moreover, we have  $u_n \rightarrow 0$  in  $X$ . We set  $v_n = \frac{u_n}{\|u_n\|}$  and assume that  $v_n \rightarrow v_0$  in  $X$  and  $v_n \rightarrow v_0$  in  $L^p(\Omega)$  and  $L^q(\Omega)$ . Given  $\phi \in X$  we have

$$\int_{\Omega} (\nabla u_n \nabla \phi - a u_n^{p-1} \phi - \lambda_n b u_n^{q-1} \phi) = 0. \quad (3.2)$$

Hence  $\lambda_n \int_{\Omega} b v_n^{q-1} \phi \rightarrow 0$ , and consequently  $\int_{\Omega} b v_0^{q-1} \phi = 0$ . Since this holds for any  $\phi \in X$ , we deduce that  $b v_0 \equiv 0$ . Taking  $\phi = v_0$  in (3.2) we obtain

$$\int_{\Omega} \nabla u_n \nabla v_0 = \int_{\Omega} a u_n^{p-1} v_0.$$

It follows that  $\int_{\Omega} \nabla v_n \nabla v_0 \rightarrow 0$ , so  $\int_{\Omega} |\nabla v_0|^2 = 0$  i.e.  $v_0$  is a constant. Therefore  $v_0 \equiv 0$ . Now, since  $u_n \in B_0^-$  for  $n$  large enough and  $N_{\lambda_n}^+ \subset B^+$ , we have  $u_n \in N_{\lambda_n}^- \cup N_{\lambda_n}^0$ , i.e.

$$E(u_n) \leq \frac{p-q}{2-q} A(u_n)$$

for  $n$  large enough. We infer that  $\limsup E(v_n) \leq 0$ , so that  $v_n \rightarrow 0$ , which contradicts  $\|v_n\| = 1$ . The proof is now complete.  $\square$

#### 4. NONEXISTENCE RESULTS

**Proposition 4.1.** *Assume  $\Omega_+^a \cap \Omega_+^b \neq \emptyset$ . Then there exists  $\bar{\lambda} > 0$  such that  $(P_{\lambda})$  has no positive solution for  $\lambda > \bar{\lambda}$ .*

*Proof.* Assume that  $(P_{\lambda})$  has a solution  $u \geq 0$ . By continuity, there exists  $\delta > 0$  and a ball  $D \subset \Omega$  such that  $a, b \geq \delta > 0$  in  $D$ . Let  $\varphi > 0$  be an eigenfunction associated to  $\lambda'_1 = \lambda_1(D)$ , i.e.  $\varphi$  is a solution of  $-\Delta \varphi = \lambda_1 \varphi$  in  $D$ ,  $\varphi = 0$  on  $\partial D$ . Then

$$\int_D \nabla \varphi \nabla u = \lambda'_1 \int_D \varphi u + \int_{\partial D} \frac{\partial \varphi}{\partial \mathbf{n}} u.$$

On the other hand, extending  $\varphi$  by zero to  $\Omega$  and using it as test function in  $(P_{\lambda})$ , we have

$$\int_{\Omega} \nabla \varphi \nabla u = \int_{\Omega} (au^{p-1} + \lambda bu^{q-1}) \varphi.$$

Hence

$$0 \geq \int_{\partial D} \frac{\partial \varphi}{\partial \mathbf{n}} u = \int_D (au^{p-1} + \lambda bu^{q-1} - \lambda'_1 u) \varphi \geq \int_D (\delta u^{p-1} + \lambda \delta u^{q-1} - \lambda'_1 u) \varphi$$

But for  $\lambda$  large enough we have  $\delta s^{p-1} + \lambda \delta s^{q-1} - \lambda'_1 s \geq 0$  for every  $s \geq 0$ . Therefore for such  $\lambda$  we must have  $u \equiv 0$ .  $\square$

**Proposition 4.2.** *Let  $\lambda > 0$ . Then the following two assertions hold:*

- (1) *Assume  $b \geq 0$  and  $\int_{\Omega} a \geq 0$ . Then  $(P_{\lambda})$  has no nontrivial non-negative solution.*
- (2) *Assume that  $b$  changes sign,  $\Omega_+^b$  is a subdomain of  $\Omega$ , and  $\Omega_-^b = \Omega \setminus \overline{\Omega_+^b}$ . If  $a \geq 0$  and  $\int_{\Omega} b \geq 0$  then  $(P_{\lambda})$  has no non-negative solution taking positive values somewhere in  $\Omega_+^b$ .*

*Proof.*

- (1) Let  $u \geq 0$  be a nontrivial solution of  $(P_{\lambda})$ . Since  $b \geq 0$ , by the strong maximum principle, we have  $u > 0$  on  $\overline{\Omega}$ . Thus we may take  $u^{1-p}$  as test function to get

$$(1-p) \int_{\Omega} |\nabla u|^2 (u + \varepsilon)^{-p} - \int_{\Omega} a - \lambda \int_{\Omega} bu^{q-p} = 0.$$

Hence

$$\int_{\Omega} a < -\lambda \int_{\Omega} bu^{q-p} < 0.$$



- (2) Let  $u \geq 0$  be a solution of  $(P_\lambda)$  such that  $u(x_0) > 0$  for some  $x_0 \in \Omega_+^b$ . Since  $\Omega_+^b$  is a subdomain, by the strong maximum principle we have  $u > 0$  in  $\Omega_+^b$ . Given  $\varepsilon > 0$ , we take  $w = (u + \varepsilon)^{1-q}$  to get

$$(1-q) \int_{\Omega} |\nabla u|^2 (u + \varepsilon)^{-q} - \int_{\Omega} a u^{p-1} (u + \varepsilon)^{1-q} - \lambda \int_{\Omega} b \left( \frac{u}{u + \varepsilon} \right)^{q-1} = 0.$$

Since  $q > 1$  we obtain

$$\lambda \int_{\Gamma_u} b \left( \frac{u}{u + \varepsilon} \right)^{q-1} < - \int_{\Omega} a u^{p-1} (u + \varepsilon)^{1-q},$$

where  $\Gamma_u = \text{supp } u$ . Letting  $\varepsilon \rightarrow 0$  and using the Lebesgue dominated convergence theorem, we get

$$\lambda \int_{\Gamma_u} b \leq - \int_{\Omega} a u^{p-q}.$$

Now, since  $b < 0$  in  $\Omega \setminus \Gamma_u$  we have

$$\int_{\Omega} b = \int_{\Gamma_u} b + \int_{\Omega \setminus \Gamma_u} b < \int_{\Gamma_u} b \leq -\lambda^{-1} \int_{\Omega} a u^{p-q}.$$

and the conclusion follows.  $\square$

## 5. BIFURCATION FOR A REGULARIZED PROBLEM

In this section we deal with the following Neumann boundary value problem with  $\lambda \in \mathbb{R}$  and  $\epsilon > 0$ :

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u + \lambda m(x)|u + \epsilon|^{q-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

Here  $m \in C^\alpha(\overline{\Omega})$ ,  $\alpha \in (0, 1)$ , satisfies

$$\Omega_+^m \neq \emptyset, \quad \text{and} \quad \int_{\Omega} m < 0. \quad (5.2)$$

Linearizing (5.1) at  $u = 0$  we obtain

$$\begin{cases} -\Delta \varphi = \lambda m \epsilon^{q-2} \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.3)$$

Under (5.2) this problem has exactly two principal eigenvalues  $\lambda = 0$  and  $\lambda = \lambda_{m,\epsilon} > 0$ , which are both simple. We denote by  $\varphi_{m,\epsilon}$  a positive eigenfunction associated to  $\lambda_{m,\epsilon}$  which is normalized as  $\|\varphi_{m,\epsilon}\|_{C(\overline{\Omega})} = 1$ . Note that  $\varphi_{m,\epsilon} > 0$  on  $\overline{\Omega}$ .

We state now the main result of this section for (5.1):

**Theorem 5.1.** *Let  $1 < q < 2 < p$  and  $0 < \epsilon \leq 1$ . Assume (5.2). Then (5.1) possesses exactly two bifurcation points  $(0, 0)$ ,  $(\lambda_{m,\epsilon}, 0)$  on  $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$  from which emanate two subcontinua of positive solutions  $\mathcal{C}_0 = \mathcal{C}_0(m, \epsilon)$ ,  $\mathcal{C}_1 = \mathcal{C}_1(m, \epsilon)$ , respectively. Moreover, the following assertions hold:*

- (1) Let  $Z$  be any complement of  $\langle 1 \rangle$  in  $C^{2+\alpha}(\overline{\Omega})$ . Then the set  $\{(\lambda, u)\}$  of nontrivial solutions of (5.1) around  $(0, 0)$  is parametrized as

$$(\lambda, u) = (\mu(s), s(1 + z(s))).$$

with  $s \in (-s_0, s_0)$ , for some  $s_0 > 0$ . Here  $\mu : (-s_0, s_0) \rightarrow \mathbb{R}$  and  $z : (-s_0, s_0) \rightarrow Z$  are continuous and satisfy  $\mu(0) = z(0) = 0$ . So  $\mathcal{C}_0$  is described exactly by  $\{(\mu(s), s(1 + z(s))) : s \in [0, s_0)\}$  around  $(0, 0)$ . Furthermore:

(a)  $\mu(s)$  satisfies

$$\lim_{s \rightarrow 0} \frac{\mu(s)}{s^{p-2}} = -\epsilon^{2-q} \frac{\int_{\Omega} a}{\int_{\Omega} m}; \quad (5.4)$$

(b) If, in addition,  $p > 2$  is an integer and  $\int_{\Omega} a = 0$ , then  $\mu(s)$  is analytic at  $s = 0$ , and its derivatives  $\mu^{(k)}$  satisfy

$$\mu^{(k)}(0) = 0 < \mu^{(2p-4)}(0) \quad \text{for } 1 \leq k < 2p - 4. \quad (5.5)$$

- (2) Let  $W$  be any complement of  $\langle \varphi_{m,\epsilon} \rangle$  in  $C^{2+\alpha}(\overline{\Omega})$ . Then the set  $\{(\lambda, u)\}$  of nontrivial solutions of (5.1) around  $(\lambda_{m,\epsilon}, 0)$  is parametrized as

$$(\lambda, u) = (\gamma(s), s(\varphi_{m,\epsilon} + w(s))),$$

with  $s \in (-s_0, s_0)$ , for some  $s_0 > 0$ . Here  $\gamma : (-s_0, s_0) \rightarrow \mathbb{R}$  and  $w : (-s_0, s_0) \rightarrow W$  are continuous and satisfy  $\gamma(0) = \lambda_{m,\epsilon}$  and  $w(0) = 0$ . So  $\mathcal{C}_1$  is described exactly by  $\{(\gamma(s), s(\varphi_{m,\epsilon} + w(s))) : s \in [0, s_0)\}$  around  $(\lambda_{m,\epsilon}, 0)$ .

- (3) Regarding the global nature of  $\mathcal{C}_0$  and  $\mathcal{C}_1$ , we have the following:

(a)  $\mathcal{C}_0 \cup \mathcal{C}_1$  does not meet  $(\lambda, 0)$  except  $\lambda = 0$  and  $\lambda = \lambda_{m,\epsilon}$ .

(b) The following alternative holds: either  $\mathcal{C}_0 = \{(\lambda, u)\}$  and  $\mathcal{C}_1 = \{(\lambda, u)\}$  are both unbounded in  $\mathbb{R} \times C(\overline{\Omega})$  or they coincide.

*Proof of Theorem 5.1.* Under (5.2) we observe that the principal eigenvalues  $\lambda = 0$  and  $\lambda = \lambda_{m,\epsilon}$  both satisfy the *transversality condition* of Crandall and Rabinowitz. Hence the standard local bifurcation theory [10, Theorem 1.7] and the unilateral global bifurcation theory [24, Theorem 1.27] (see also [20, Theorem 6.4.3]) are applicable at  $(0, 0)$  and  $(\lambda_{m,\epsilon}, 0)$ . We obtain then two subcontinua  $\mathcal{C}_0, \mathcal{C}_1$  of positive solutions of (5.1) emanating from  $(0, 0)$  and  $(\lambda_{m,\epsilon}, 0)$ , respectively. Moreover, assertions (1), (2) and (3) are promptly verified, except (5.4) and (5.5).

Let us show (5.4) in assertion (1)(a). From assertion (1) we deduce that

$$\int_{\Omega} \{a(s + sz)^{p-1} + \mu m(s + sz + \epsilon)^{q-2}(s + sz)\} = 0.$$

It follows that

$$\frac{\mu(s)}{s^{p-2}} = -\frac{\int_{\Omega} a(1 + z)^{p-2}}{\int_{\Omega} m(s + sz + \epsilon)^{q-2}(1 + z)} \rightarrow -\frac{\int_{\Omega} a}{\int_{\Omega} m\epsilon^{q-2}}, \quad \text{as } s \rightarrow 0^+,$$

as desired.

Next we verify (5.5) in assertion (1)(b). Following the Lyapunov-Schmidt method, we reduce (5.1) to a bifurcation equation around the origin in  $\mathbb{R}^2$ . Let  $w = Qu = u - \frac{1}{\Omega} \int_{\Omega} u$ , where  $Q$  is defined as a linear mapping from  $L^2(\Omega)$  to  $\{w \in L^2(\Omega) : \int_{\Omega} w = 0\}$ . We also write  $t = \frac{1}{|\Omega|} \int_{\Omega} u$ , so that  $u = t + w$ . Using  $Q$  we decompose (5.1) orthogonally in the

following way: for  $|\lambda| < \lambda^*$  and  $u \in U$ , a small neighborhood of 0 in  $\mathcal{C}^{2+\alpha}(\overline{\Omega})$ , we have

$$Q(-\Delta u) = Q(au^{p-1} + \lambda m(u + \epsilon)^{q-2}u), \quad (5.6)$$

$$(1 - Q)(-\Delta u) = (1 - Q)(au^{p-1} + \lambda m(u + \epsilon)^{q-2}u). \quad (5.7)$$

By applying the implicit function theorem we see that (5.6) is uniquely solvable at  $(\lambda, t, w) = (0, 0, 0)$  by some  $w = w(\lambda, t)$  which is analytic at  $(0, 0)$  and satisfies  $w(\lambda, 0) = 0$  for  $\lambda$  sufficiently small. We plug  $u = t + w(\lambda, t)$  in (5.7), to obtain the following bifurcation equation around the origin in  $\mathbb{R}^2$ :

$$\Phi(\lambda, t) := \int_{\Omega} a(t + w(\lambda, t))^{p-1} + \lambda \int_{\Omega} m(t + w(\lambda, t) + \epsilon)^{q-2}(t + w(\lambda, t)) = 0. \quad (5.8)$$

Note that  $\Phi$  is also analytic at  $(0, 0)$ .

We shall now analyse  $\Phi$  in (5.8) for  $(\lambda, t)$  around  $(0, 0)$  using its Taylor series expansion. As a preliminary, we compute the partial derivatives of  $w(\lambda, t)$  at  $(0, 0)$ :

**Lemma 5.2.**

- (1)  $\frac{\partial^k w}{\partial \lambda^k}(0, 0) = 0$  for every  $k \geq 0$ .
- (2)  $\frac{\partial w}{\partial t}(0, 0) = 0$ .
- (3)  $\frac{\partial^2 w}{\partial t \partial \lambda}(0, 0) (= \frac{\partial^2 w}{\partial \lambda \partial t}(0, 0))$  is a unique solution of the problem

$$\begin{cases} -\Delta w = \epsilon^{q-2}Q[m] & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} w = 0. \end{cases}$$

- (4) For every integer  $k \geq 2$  we have

$$\frac{\partial^k w}{\partial t^k}(0, 0) = \begin{cases} w_{k,a}, & p = k + 1, \\ 0, & p > k + 1, \end{cases}$$

where  $w_{k,a}$  is the unique solution of the problem

$$\begin{cases} -\Delta w = (k!)a & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} w = 0. \end{cases}$$

**Remark 5.3.** Lemma 5.2(2)(4) tells us that

$$\begin{cases} \frac{\partial^j w}{\partial t^j}(0, 0) = 0, & 1 \leq j < p - 1, \\ \frac{\partial^{p-1} w}{\partial t^{p-1}}(0, 0) = w_{p-1,a}. \end{cases}$$

*Proof of Lemma 5.2.* We denote the partial derivatives of  $w$  simply by  $w_{\lambda}, w_t, w_{\lambda\lambda}, w_{tt}, w_{\lambda t}$ .

- (1) By the uniqueness ensured by the implicit function theorem, we see that  $w(\lambda, 0) = 0$  for  $\lambda$  close to 0. This provides assertion (1).
- (2) Note that  $w = w(\lambda, t)$  satisfies

$$\begin{cases} -\Delta w = Q[a(t + w)^{p-1} + \lambda m(t + w + \epsilon)^{q-2}(t + w)] & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Differentiating this problem with respect to  $t$  we obtain

$$\begin{cases} -\Delta w_t = Q[a(p-1)(t+w)^{p-2}(1+w_t) \\ \quad + \lambda m\{(q-2)(t+w+\epsilon)^{q-3}(1+w_t)(t+w) \\ \quad + (t+w+\epsilon)^{q-2}(1+w_t)\}] & \text{in } \Omega, \\ \frac{\partial w_t}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.9)$$

Putting  $(\lambda, t) = (0, 0)$  here we deduce

$$-\Delta w_t(0, 0) = 0 \quad \text{in } \Omega, \quad \int_{\Omega} w_t = 0,$$

which yields Assertion (2).

(3) Differentiating (5.9) with respect to  $\lambda$  we get

$$\begin{cases} -\Delta w_{t\lambda} = Q[a(p-1)\{(p-2)(t+w)^{p-3}w_{t\lambda}(1+w_t) + (t+w)^{p-2}w_{t\lambda}\} \\ \quad + m\{(q-2)(t+w+\epsilon)^{q-3}(1+w_t)(t+w) + (t+w+\epsilon)^{q-2}(1+w_t)\} \\ \quad + \lambda m\{(q-2)(q-3)(t+w+\epsilon)^{q-4}w_{t\lambda}(1+w_t)(t+w) \\ \quad + (q-2)(t+w+\epsilon)^{q-3}w_{t\lambda}(t+w) \\ \quad + (q-2)(t+w+\epsilon)^{q-3}(1+w_t)w_{t\lambda} \\ \quad + (q-2)(t+w+\epsilon)^{q-3}w_{t\lambda}(1+w_t) \\ \quad + (t+w+\epsilon)^{q-2}w_{t\lambda}\}] & \text{in } \Omega, \\ \frac{\partial w_{t\lambda}}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.10)$$

Putting  $(\lambda, t) = (0, 0)$  here we deduce

$$-\Delta w_{t\lambda}(0, 0) = Q[m\epsilon^{q-2}] \quad \text{in } \Omega, \quad \int_{\Omega} w_{t\lambda} = 0,$$

which yields assertion (3).

(4) We consider  $\frac{\partial^k w}{\partial t^k}(0, 0)$  for  $k \geq 2$ . We shall differentiate (5.9) with respect to  $t$  repeatedly and put  $(\lambda, t) = (0, 0)$  therein. To this end it is enough to consider the first term on the right-hand side of the equation in (5.9), i.e. the term  $\eta := Q[a(p-1)(t+w)^{p-2}(1+w_t)]$ . For instance, let us discuss the case  $k = 2$ : we consider the derivative

$$\eta_t = Q[a(p-1)(p-2)(t+w)^{p-3}(1+w_t) + a(p-1)(t+w)^{p-2}w_{tt}].$$

It follows that

$$\eta_t(0, 0) = \begin{cases} Q[(p-1)(p-2)a], & \text{if } p = 3, \\ 0, & \text{if } p > 3. \end{cases}$$

More generally, putting  $\eta = Q[a(p-1)(t+w)^{p-2}(1+w_t)]$  we obtain

$$\frac{\partial^{k-1} \eta}{\partial t^{k-1}}(0, 0) = \begin{cases} Q[(p-1)(p-2) \cdots (p-k)a], & \text{if } p = k+1, \\ 0, & \text{if } p > k+1. \end{cases}$$

Since  $\int_{\Omega} a = 0$ , it follows that

$$-\Delta \frac{\partial^k w}{\partial t^k}(0, 0) = \begin{cases} (p-1)(p-2) \cdots (p-k)a, & \text{if } p = k+1, \\ 0, & \text{if } p > k+1, \end{cases}$$

which yields assertion (4).

The proof of Lemma 5.2 is now complete.  $\square$

Using Lemma 5.2, we obtain the partial derivatives of  $\Phi$  at  $(0, 0)$ :

**Lemma 5.4.**

- (1)  $\frac{\partial^k \Phi}{\partial \lambda^k}(0, 0) = 0$  for every integer  $k \geq 0$ .
- (2)  $\frac{\partial \Phi}{\partial t}(0, 0) = \frac{\partial^2 \Phi}{\partial t^2}(0, 0) = 0$ .
- (3)  $\frac{\partial^2 \Phi}{\partial t \partial \lambda}(0, 0) = \frac{\partial^2 \Phi}{\partial \lambda \partial t}(0, 0) = \epsilon^{q-2} \int_{\Omega} m < 0$ .
- (4) For every integer  $k \geq 2$  we have

$$\frac{\partial^{2k-1} \Phi}{\partial t^{2k-1}}(0, 0) = \begin{cases} C_k \int_{\Omega} \left| \nabla \frac{\partial^k w}{\partial t^k}(0, 0) \right|^2, & \text{if } p = k + 1, \\ 0, & \text{if } p > k + 1 \end{cases} \quad (5.11)$$

for some constant  $C_k > 0$ , and

$$\frac{\partial^{2k} \Phi}{\partial t^{2k}}(0, 0) = 0, \quad \text{if } p \geq k + 2. \quad (5.12)$$

**Remark 5.5.** Lemma 5.4(4) tells us that

$$\begin{cases} \frac{\partial^j \Phi}{\partial t^j}(0, 0) = 0, & \text{if } 3 \leq j < 2p - 3, \\ \frac{\partial^{2p-3} \Phi}{\partial t^{2p-3}}(0, 0) > 0. \end{cases}$$

*Proof of Lemma 5.4.* We denote by  $\Phi_{\lambda}, \Phi_t, \Phi_{\lambda\lambda}, \Phi_{\lambda t}$  the derivatives of  $\Phi$ .

- (1) It is straightforward from assertion (1) in Lemma 5.2.
- (2) Differentiating (5.8) with respect to  $t$  we obtain

$$\begin{aligned} \Phi_t &= \int_{\Omega} a(p-1)(t+w)^{p-2}(1+w_t) \\ &+ \lambda \int_{\Omega} m\{(q-2)(t+w+\epsilon)^{q-3}(1+w_t)(t+w) + (t+w+\epsilon)^{q-2}(1+w_t)\} \end{aligned} \quad (5.13)$$

It follows that  $\Phi_t(0, 0) = 0$ . Once again we differentiate (5.13) with respect to  $t$  to obtain

$$\begin{aligned} \Phi_{tt} &= \int_{\Omega} a(p-1)\{(p-2)(t+w)^{p-3}(1+w_t)^2 + (t+w)^{p-2}w_{tt}\} \\ &+ \lambda h(\lambda, t), \end{aligned} \quad (5.14)$$

where  $h$  is bounded. It follows that  $\Phi_{tt}(0, 0) = 0$  when  $p > 3$ . In addition, this remains true if  $p = 3$  since  $\int_{\Omega} a = 0$ . Assertion (2) is then proved.

- (3) Differentiating (5.13) with respect to  $\lambda$  we get

$$\begin{aligned} \Phi_{t\lambda} &= \int_{\Omega} a(p-1)\{(p-2)(t+w)^{p-3}w_{\lambda}(1+w_t) + (t+w)^{p-2}w_{t\lambda}\} \\ &+ \int_{\Omega} m\{(q-2)(t+w+\epsilon)^{q-3}(1+w_t)(t+w) + (t+w+\epsilon)^{q-2}(1+w_t)\} \\ &+ \lambda \int_{\Omega} m\{(q-2)(q-3)(t+w+\epsilon)^{q-4}w_{\lambda}(1+w_t)(t+w) \\ &\quad + (q-2)(t+w+\epsilon)^{q-3}w_{t\lambda}(t+w) \\ &\quad + (q-2)(t+w+\epsilon)^{q-3}(1+w_t)w_{\lambda} \\ &\quad + (q-2)(t+w+\epsilon)^{q-3}w_{\lambda}(1+w_t) \\ &\quad + (t+w+\epsilon)^{q-2}w_{t\lambda}\}. \end{aligned}$$

Putting  $(\lambda, t) = (0, 0)$  here, it follows that

$$\Phi_{t\lambda}(0, 0) = \int_{\Omega} m \epsilon^{q-2},$$

which yields Assertion (3).

(4) In (5.14) we put

$$\zeta(\lambda, t) = \int_{\Omega} a(p-1) \{ (p-2)(t+w)^{p-3}(1+w_t)^2 + (t+w)^{p-2}w_{tt} \}$$

Then we deduce from (5.14) that

$$\frac{\partial^j \Phi}{\partial t^j} = \frac{\partial^{j-2} \zeta}{\partial t^{j-2}} + \lambda H_j(\lambda, t), \quad (5.15)$$

where  $H_j$  is bounded. Let us verify (5.11) in the case  $k = 2$ : we consider  $j = 3$  in (5.15) and observe that

$$\begin{aligned} \zeta_t = \int_{\Omega} a(p-1) \{ (p-2)(p-3)(t+w)^{p-4}(1+w_t)^3 \\ + 3(p-2)(t+w)^{p-3}(1+w_t)w_{tt} + (t+w)^{p-2}w_{ttt} \}. \end{aligned} \quad (5.16)$$

Here it is understood that  $(t+w)^\ell = 0$  if  $\ell < 0$ . Putting  $(\lambda, t) = (0, 0)$  here, it follows from Lemma 5.2(4) that

$$\Phi_{ttt}(0, 0) = \zeta_t(0, 0) = \begin{cases} 6 \int_{\Omega} a w_{tt}(0, 0) = 3 \int_{\Omega} |\nabla w_{tt}(0, 0)|^2, & \text{if } p = 3, \\ 0, & \text{if } p > 3. \end{cases}$$

Here we have used that  $\int_{\Omega} a = 0$  and  $(t+w)^\ell = 0$  at  $(\lambda, t) = (0, 0)$  for some integer  $\ell \geq 1$ . Thus Assertion (5.11) with  $k = 2$  has been verified.

We prove now (5.12) in the case  $k = 2$ : we consider  $j = 4$  in (5.15) and differentiate (5.16) with respect to  $t$  once more to obtain

$$\begin{aligned} \zeta_{tt} = \int_{\Omega} a(p-1) \{ (p-2)(p-3)(p-4)(t+w)^{p-5}(1+w_t)^4 \\ + 6(p-2)(p-3)(t+w)^{p-4}(1+w_t)^2w_{tt} \\ + 3(p-2)(t+w)^{p-3}w_{tt}^2 + 4(p-2)(t+w)^{p-3}(1+w_t)w_{ttt} \\ + (t+w)^{p-2}w_{tttt} \} \end{aligned}$$

When  $p \geq 4$  we deduce  $\zeta_{tt}(0, 0) = 0$  in view of Remark 5.3. Hence Assertion (5.12) with  $k = 2$  has been verified. In a similar way, we can verify assertions (5.11) and (5.12) for the general case  $k > 2$ , using the differential chain rule.

The proof of Lemma 5.4 is now complete.  $\square$

We conclude now the verification of (5.5): from Lemma 5.4(1)-(3) and the fact that  $\Phi$  is analytic at  $(0, 0)$ , we deduce that the Taylor series expansion of  $\Phi$  at  $(0, 0)$  is provided by

$$\Phi(\lambda, t) = t \left( \lambda \frac{\partial^2 \Phi}{\partial t \partial \lambda}(0, 0) + \Psi(\lambda, t) \right)$$

where  $\Psi(\lambda, t)$  is a higher order term. We put

$$\xi(\lambda, t) := \lambda \frac{\partial^2 \Phi}{\partial t \partial \lambda}(0, 0) + \Psi(\lambda, t).$$

Note that  $\xi(0, 0) = 0$  and

$$\frac{\partial \xi}{\partial \lambda}(0, 0) = \frac{\partial^2 \Phi}{\partial t \partial \lambda}(0, 0) < 0.$$

Hence the implicit function theorem can be applied to deduce that the solution set of  $\xi(\lambda, t) = 0$  near  $(0, 0)$  is explicitly given by a function  $\lambda(t)$  satisfying  $\lambda(0) = 0$ .

We see that  $\lambda'(0) = -\frac{\frac{\partial \xi}{\partial t}(0, 0)}{\frac{\partial \xi}{\partial \lambda}(0, 0)} = 0$ , since  $\frac{\partial \xi}{\partial t}(0, 0) = \frac{\partial^2 \Phi}{\partial t^2}(0, 0) = 0$  from Lemma 5.4(2).

However, since  $\frac{\partial^j \xi}{\partial t^j}(0, 0) = \frac{\partial^{j+1} \Phi}{\partial t^{j+1}}(0, 0)$ , Remark 5.5 provides that

$$\begin{cases} \frac{\partial^j \xi}{\partial t^j}(0, 0) = 0 \text{ if } 2 \leq j < 2p - 4, \\ \frac{\partial^{2p-4} \xi}{\partial t^{2p-4}}(0, 0) > 0. \end{cases}$$

This implies that

$$\begin{cases} \lambda^{(j)}(0) = 0 \text{ if } 1 \leq j < 2p - 4, \\ \lambda^{(2p-4)}(0) = -\frac{\frac{\partial^{2p-4} \xi}{\partial t^{2p-4}}(0, 0)}{\frac{\partial \xi}{\partial \lambda}(0, 0)} > 0. \end{cases}$$

Assertion (5.5) is now proved.

The proof of Theorem 5.1 is now complete.  $\square$

**Remark 5.6.** As pointed out in Remark 1.2(1), assertion (1)(b) with  $\int_{\Omega} b > 0$  and assertion (2)(a) in Theorem 1.1 hold true without the restriction  $p < \frac{2N}{N-2}$ ,  $N > 2$ . Indeed, this is verified by a rescaling argument for  $(P_{\lambda})$  with  $v = \lambda^{-\frac{1}{p-q}} u$ , the Lyapunov-Schmidt reduction of the rescaled problem developed in this section, and an application of the implicit function theorem to the reduced problem. More precisely, by the rescaling  $v = \lambda^{-\frac{1}{p-q}} u$  with  $\lambda > 0$  and  $u \geq 0$ , we reduce  $(P_{\lambda})$  to the problem (see (2.10)).

$$\begin{cases} -\Delta v = \mu(av^{p-1} + bv^{q-1}) & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \quad (5.17)$$

with  $\mu = \lambda^{\frac{p-2}{p-q}}$ . Employing the Lyapunov-Schmidt method with the linear mapping  $w = Qv = v - \frac{1}{|\Omega|} \int_{\Omega} v$  as in (5.6) and (5.7), we have the following bifurcation equation in  $\mathbb{R}^2$ :

$$\begin{cases} \Phi(\mu, t) = \int_{\Omega} \{a(t + w(\mu, t))^{p-1} + b(t + w(\mu, t))^{q-1}\} = 0, & (\mu, t) \simeq (0, c^*), \\ \Phi(0, c^*) = 0. \end{cases} \quad (5.18)$$

Here  $t = \frac{1}{|\Omega|} \int_{\Omega} v$  and  $w = w(\mu, t)$  is the unique solution in  $C^{2+\alpha}(\overline{\Omega})$  of the following boundary value problem defined in a neighborhood of  $(\mu, t, w) = (0, c^*, 0)$ :

$$\begin{cases} -\Delta w = \mu Q(a(t + w)^{p-1} + b(t + w)^{q-1}) & \text{in } \Omega, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness of  $w(\mu, t)$  is ensured by the implicit function theorem. Then we can prove that if  $(\int_{\Omega} a)(\int_{\Omega} b) < 0$  then

$$\Phi_t(0, c^*) = (p-1)(c^*)^{p-2} \left( \int_{\Omega} a \right) + (q-1)(c^*)^{q-2} \left( \int_{\Omega} b \right) = (q-p)(c^*)^{q-2} \left( \int_{\Omega} b \right) \neq 0.$$

Still by the implicit function theorem, there exists a unique solution  $t(\mu)$  of (5.18), i.e.

$$(5.18) \iff \begin{cases} t = t(\mu) \text{ for } \mu \simeq 0, \\ t(0) = c^*. \end{cases}$$

Hence (5.17) has a positive solution  $\hat{v}_\lambda = v_\mu = t(\mu) + w(\mu, t(\mu))$  with  $\mu = \lambda^{\frac{p-2}{p-q}}$  bifurcating to the region  $\lambda > 0$  from  $\{(0, c) : c \text{ is a constant}\}$  at  $(0, c^*)$ , i.e. satisfying  $\hat{v}_0 = c^*$ . Moreover, this solution is unique and  $\hat{v}_\lambda \rightarrow c^*$  in  $C^{2+\alpha}(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ . Finally, going back to  $(P_\lambda)$  by the rescaling  $u = \lambda^{\frac{1}{p-q}} v$ , we have a positive solution  $u_\lambda = \lambda^{\frac{1}{p-q}} \hat{v}_\lambda$  of  $(P_\lambda)$  and  $\lambda^{-\frac{1}{p-q}} u_\lambda \rightarrow c^*$  in  $C^{2+\alpha}(\overline{\Omega})$  as  $\lambda \rightarrow 0^+$ , as desired.

As a byproduct relying on the uniqueness result for  $\hat{v}_\lambda$  in Remark 5.6, we show that the variational positive solution  $u_{1,\lambda}$  of  $(P_\lambda)$  given by Proposition 2.6 is asymptotically stable for  $\lambda > 0$  sufficiently small when  $\int_\Omega a < 0 < \int_\Omega b$ .

**Proposition 5.7.** *Under the assumptions of Proposition 2.6, if  $\int_\Omega a < 0 < \int_\Omega b$  then  $u_{1,\lambda}$  is asymptotically stable for  $\lambda > 0$  sufficiently small.*

*Proof.* With the aid of the argument in Remark 5.6, it suffices to show that the positive solution  $v_\mu = t(\mu) + w(\mu, t(\mu))$  of (5.17) is asymptotically stable for  $\mu > 0$  sufficiently small. To this end, recalling (1.7), we investigate the sign of the first eigenvalue  $\hat{\gamma} = \hat{\gamma}_{1,\mu}$  of the following eigenvalue problem to discuss the linearized stability of  $v_\mu$ .

$$\begin{cases} -\Delta \phi = \mu ((p-1)av_\mu^{p-2} + (q-1)bv_\mu^{q-2})\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

By  $\hat{\phi} = \hat{\phi}_{1,\mu}$  we denote a positive eigenfunction associated with  $\hat{\gamma}_{1,\mu}$  and normalized as  $\int_\Omega \hat{\phi}_{1,\mu}^2 = 1$ . Note that  $\hat{\gamma}_{1,0} = 0$  and  $\hat{\phi}_{1,0} = |\Omega|^{-\frac{1}{2}}$ , and moreover that the mapping  $\mu \mapsto (\hat{\gamma}_{1,\mu}, \hat{\phi}_{1,\mu})$  is continuous in  $\mathbb{R} \times C^{2+\alpha}(\overline{\Omega})$  for  $\mu$  close to 0 by the implicit function theorem.

We consider  $\int_\Omega (-\Delta v_\mu) \frac{\hat{\phi}^2}{v_\mu}$ . By the divergence theorem we have

$$\hat{\gamma} = \left( \int_\Omega \left| \frac{\hat{\phi}}{v_\mu} \nabla v_\mu - \nabla \hat{\phi} \right|^2 \right) + \mu M_\mu \geq \mu M_\mu, \quad (5.19)$$

where  $M_\mu = \int_\Omega (-(p-2)av_\mu^{p-2} + (2-q)bv_\mu^{q-2}) \hat{\phi}^2$ . Since  $v_\mu \rightarrow c^* = \left( \frac{\int_\Omega b}{-\int_\Omega a} \right)^{\frac{1}{p-q}}$  in  $C(\overline{\Omega})$  as  $\mu \rightarrow 0$ , we deduce that

$$M_\mu \longrightarrow \frac{(p-q)}{|\Omega|} \left( \int_\Omega b \right)^{\frac{p-2}{p-q}} \left( -\int_\Omega a \right)^{\frac{2-q}{p-q}} > 0 \quad \text{as } \mu \rightarrow 0.$$

Hence it follows from (5.19) that  $\hat{\gamma} > 0$  for  $\mu > 0$  sufficiently small, as desired.  $\square$

## 6. EXISTENCE OF LOOP TYPE SUBCONTINUA: RESULTS AND EXPECTATIONS

Let  $\epsilon > 0$ , and let  $b_\epsilon = b - \epsilon$ . Then we consider the following regularized problem for  $(P_\lambda)$ .

$$(P_{\lambda,\epsilon}) \quad \begin{cases} -\Delta u = \lambda b_\epsilon(x)|u + \epsilon|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

First we establish an *a priori* bound on  $|\lambda|$  for nontrivial non-negative solutions of  $(P_{\lambda,\epsilon})$ .



**Proposition 6.1.** *Assume that there exists a ball  $B$  such that  $\overline{B} \subset \Omega$  with the condition*

$$a \geq 0, \quad a \not\equiv 0 \quad \text{and} \quad b > 0 \quad \text{on} \quad \overline{B}.$$

*Then there exists constants  $\overline{\lambda} > 0$  and  $\epsilon_0 > 0$  such that  $(P_{\lambda, \epsilon})$  has no nontrivial non-negative solutions for any  $\lambda \geq \overline{\lambda}$  and  $\epsilon \in (0, \epsilon_0]$ .*

*Proof.* The proof is carried out as for Proposition 4.1, with small modifications. Let  $\varphi = \varphi_D(B) \in C^2(\overline{B})$  be a positive eigenfunction associated with the first eigenvalue  $\lambda_1 = \lambda_D(B) > 0$  of (2.9) with  $\Omega$  replaced by  $B$ . We extend  $\varphi$  to the whole  $\Omega$  by setting  $\varphi = 0$  in  $\Omega \setminus \overline{B}$ . Then  $\varphi \in H^1(\Omega)$ .

Let  $u \in C^2(\overline{\Omega})$  be a nontrivial non-negative solution of  $(P_{\lambda, \epsilon})$ . Note that  $u > 0$  in  $\overline{\Omega}$ . Let  $\epsilon_0 > 0$  be such that  $b_{\epsilon_0} > 0$  in  $\overline{B}$ . By the divergence theorem, we deduce that  $\int_B \nabla \cdot u \nabla \varphi = \int_{\partial B} u \frac{\partial \varphi}{\partial \mathbf{n}} < 0$ . It follows that

$$\int_B \nabla u \nabla \varphi - \lambda_1 \int_B a u \varphi < 0.$$

On the other hand, the function  $u$  satisfies

$$\int_{\Omega} \nabla u \nabla w - \int_{\Omega} a u^{p-1} w - \lambda \int_{\Omega} b_{\epsilon} (u + \epsilon)^{q-2} u w = 0, \quad \forall w \in H^1(\Omega).$$

Taking  $w = \varphi$ , we have

$$\int_B \nabla u \nabla \varphi = \int_B a u^{p-1} \varphi + \lambda \int_B b_{\epsilon} (u + \epsilon)^{q-2} u \varphi.$$

We deduce then that

$$\int_B u^{q-1} \varphi \left( a u^{p-q} + \lambda b_{\epsilon} \left( \frac{u}{u + \epsilon} \right)^{2-q} - \lambda_1 a u^{2-q} \right) < 0.$$

Let us set

$$h(x, s) = a(x) s^{p-q} + \lambda b_{\epsilon}(x) \left( \frac{s}{s + \epsilon} \right)^{2-q} - \lambda_1 a(x) s^{2-q}, \quad \text{for } (x, s) \in \overline{B} \times [0, \infty).$$

For our purpose it suffices to show that there exists  $\overline{\lambda}$  such that  $h \geq 0$  if  $\lambda \geq \overline{\lambda}$ ,  $(x, s) \in \overline{B} \times [0, \infty)$ , and  $\epsilon \in (0, \epsilon_0]$ . Indeed, note that

$$h(x, s) \geq a s^{p-q} - \lambda_1 a s^{2-q} = a s^{2-q} (s^{p-2} - \lambda_1) \geq 0$$

if  $s \geq s_0 := \lambda_1^{\frac{1}{p-2}}$  and  $x \in \overline{B}$ . Next we observe that

$$\frac{\left( \frac{s}{s + \epsilon} \right)^{2-q}}{s^{2-q}} = \left( \frac{1}{s + \epsilon} \right)^{2-q} \geq \left( \frac{1}{s_0 + \epsilon_0} \right)^{2-q}$$

if  $0 \leq s \leq s_0$  and  $\epsilon \in (0, \epsilon_0]$ . Hence, if  $0 \leq s \leq s_0$ ,  $x \in \overline{B}$ , and  $\epsilon \in (0, \epsilon_0]$ , then

$$h \geq \lambda b_{\epsilon} \left( \frac{s}{s + \epsilon} \right)^{2-q} - \lambda_1 a s^{2-q} \geq \left( \lambda \min_{\overline{B}} b_{\epsilon_0} \left( \frac{1}{s_0 + \epsilon_0} \right)^{2-q} - \lambda_1 \|a\|_{C(\overline{B})} \right) s^{2-q},$$

So, if in addition,

$$\lambda \geq \overline{\lambda} := \frac{\lambda_1 \|a\|_{C(\overline{B})} (s_0 + \epsilon_0)^{2-q}}{\min_{\overline{B}} b_{\epsilon_0}}$$

then  $h \geq 0$ , as desired. The proof of Proposition 6.1 is now complete.  $\square$

The following result is a direct consequence of Proposition 6.1:

**Corollary 6.2.** *Assume  $(H_0)$ . Then there exists constant  $\bar{\lambda} > 0$  and  $\epsilon_0 > 0$  such that  $(P_{\lambda,\epsilon})$  has no nontrivial non-negative solutions for any  $|\lambda| \geq \bar{\lambda}$  and  $\epsilon \in (0, \epsilon_0]$ .*

*Proof.* It suffices to note that if  $u$  is a nontrivial non-negative solution of  $(P_{\lambda,\epsilon})$  for some  $\lambda < 0$ , then  $-\Delta u = au^{p-1} + (-\lambda)(-b_\epsilon)(u + \epsilon)^{q-2}u$  in  $\Omega$ .  $\square$

Next we obtain *a priori* bounds in  $C(\bar{\Omega})$  for nontrivial non-negative solutions of  $(P_{\lambda,\epsilon})$ . We recall that

$$\Omega_\pm^a = \{x \in \Omega : a \gtrless 0\}, \quad \Omega_0^a = \{x \in \Omega : a = 0\}, \quad \Omega_\pm^b = \{x \in \Omega : b \gtrless 0\}.$$

We assume that  $\Omega_\pm^a$  are both subdomains of  $\Omega$  with smooth boundary and satisfy  $(H_1)$ , i.e.

$$\overline{\Omega_+^a} \subset \Omega, \quad \overline{\Omega_+^a} \cup \Omega_-^a = \Omega.$$

**Proposition 6.3.** *Assume  $(H_1)$  and let  $\Lambda > 0$ . Suppose there exists a constant  $C_1 > 0$  such that  $\|u\|_{C(\overline{\Omega_+^a})} \leq C_1$  for all nontrivial non-negative solutions  $u$  of  $(P_{\lambda,\epsilon})$  with  $\lambda \in [0, \Lambda]$  and  $\epsilon \in (0, 1]$ . Then there exists  $C_2 > 0$  such that  $\|u\|_{C(\overline{\Omega})} \leq C_2$  for all nontrivial non-negative solutions  $u$  of  $(P_{\lambda,\epsilon})$  with  $\lambda \in [0, \Lambda]$  and  $\epsilon \in (0, 1]$ .*

*Proof.* The argument relies on the use of the comparison principle for a concave problem: consider the problem

$$\begin{cases} -\Delta v = -a^-v^{p-1} + \lambda b^+(v + \epsilon)^{q-2}v & \text{in } \Omega_-^a, \\ v = C_1 & \text{on } \partial\Omega_+^a, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.1)$$

Let  $u$  be a nontrivial non-negative solution of  $(P_{\lambda,\epsilon})$  with  $\lambda \in [0, \Lambda]$  and  $\epsilon \in (0, 1]$ . The strong maximum principle and boundary point lemma ensure  $u > 0$  in  $\overline{\Omega_-^a}$ . Since  $u \leq C_1$  on  $\partial\Omega_-^a$  from the assumption and  $b_\epsilon < b \leq b^+$ ,  $u$  is a subsolution of (6.1), that is,

$$\begin{cases} -\Delta u \leq -a^-u^{p-1} + \lambda b^+(u + \epsilon)^{q-2}u & \text{in } \Omega_-^a, \\ u \leq C_1 & \text{on } \partial\Omega_+^a, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

We next construct a supersolution of (6.1). Consider the unique positive solution  $w_0$  of the problem

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega_-^a, \\ w = 0 & \text{on } \partial\Omega_+^a, \\ \frac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Set  $\overline{w} = C(w_0 + 1)$ ,  $C > 0$ . Then  $\overline{w} > C_1$  on  $\partial\Omega_+^a$  and  $\frac{\partial \overline{w}}{\partial \mathbf{n}} = 0$  on  $\partial\Omega$  if  $C$  is large. Moreover

$$-\Delta \overline{w} - \{-a^- \overline{w}^{p-1} + \lambda b^+(\overline{w} + \epsilon)^{q-2} \overline{w}\} \geq C \{1 - \lambda b^+ \overline{w}^{q-2} (w_0 + 1)\} > 0 \quad \text{in } \Omega_-^a.$$

Hence  $\overline{w}$  is a supersolution of (6.1), where  $C$  can be chosen independently of  $\lambda \in [0, \Lambda]$  and  $\epsilon \in (0, 1]$ . By using a variant of [3, Lemma 3.3], we deduce  $u \leq \overline{w}$  in  $\overline{\Omega_-^a}$ , so that  $u \leq C_2 := C_1 + \max_{\overline{\Omega_-^a}} \overline{w}$  in  $\overline{\Omega}$ , as desired. The proof of Proposition 6.3 is complete.  $\square$

Proposition 6.3 can be extended to  $\lambda < 0$  as follows:

**Corollary 6.4.** *Assume  $(H_1)$  and let  $\Lambda > 0$ . Suppose that there exists a constant  $C_1 > 0$  such that  $\|u\|_{C(\overline{\Omega_+^a})} \leq C_1$  for all nontrivial non-negative solutions  $u$  of  $(P_{\lambda,\epsilon})$  with  $|\lambda| \leq \Lambda$  and  $\epsilon \in (0, 1]$ . Then there exists  $C_2 > 0$  such that  $\|u\|_{C(\overline{\Omega})} \leq C_2$  for all nontrivial non-negative solutions  $u$  of  $(P_{\lambda,\epsilon})$  with  $|\lambda| \leq \Lambda$  and  $\epsilon \in (0, 1]$ .*

*Proof.* We discuss the case  $-\Lambda \leq \lambda \leq 0$ . Note that any nontrivial non-negative solution  $u$  of  $(P_{\lambda,\epsilon})$  with  $\lambda \in [-\Lambda, 0]$  and  $\epsilon \in (0, 1]$  satisfies

$$-\Delta u = au^{p-1} + (-\lambda)(-b_\epsilon)(u + \epsilon)^{q-2}u \quad \text{in } \Omega_-^a.$$

with  $-\lambda \in [0, \Lambda]$ . Instead of (6.1) we consider the following concave problem

$$\begin{cases} -\Delta v = -a^-v^{p-1} + (-\lambda)(b^- + 1)(v + \epsilon)^{q-2}v & \text{in } \Omega_-^a, \\ v = C_1 & \text{on } \partial\Omega_+^a, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then  $u$  is a subsolution of this problem. The rest of the argument is the same as in the proof of Proposition 6.3.  $\square$

Based on Corollary 6.4, we use an argument from Amann and Lopez-Gomez [2] (see also Section 6 of López-Gómez, Molina-Meyer and Tellini [22]) to obtain *a priori* bounds in  $C(\overline{\Omega})$  for positive solutions of  $(P_{\lambda,\epsilon})$ :

**Proposition 6.5.** *Assume  $(H_1)$  and  $(H_2)$ . Then, for any  $\Lambda > 0$  there exists  $C_\Lambda > 0$  such that  $\|u\|_{C(\overline{\Omega})} \leq C_\Lambda$  for all nontrivial non-negative solutions of  $(P_{\lambda,\epsilon})$  with  $\lambda \in [-\Lambda, \Lambda]$  and  $\epsilon \in (0, 1]$ .*

Now we assume (1.14). Note that  $b_\epsilon$  satisfies (5.2) with  $m = b_\epsilon$ . Then, by applying Theorem 5.1 with  $m = b_\epsilon$ ,  $(P_{\lambda,\epsilon})$  possesses exactly two bifurcation points  $(0, 0)$  and  $(\lambda_\epsilon, 0)$ , where  $\lambda_\epsilon = \lambda_{b_\epsilon, \epsilon}$ , from which there bifurcate subcontinua  $\mathcal{C}_0(\epsilon) = \mathcal{C}_0(b_\epsilon, \epsilon)$  and  $\mathcal{C}_1(\epsilon) = \mathcal{C}_1(b_\epsilon, \epsilon)$  of positive solutions, respectively, and  $\mathcal{C}_0(\epsilon)$  and  $\mathcal{C}_1(\epsilon)$  satisfy assertions (1)-(3) in Theorem 5.1 with  $m = b_\epsilon$ . Moreover, the bifurcation point  $(\lambda_\epsilon, 0)$  tends to  $(0, 0)$ :

**Lemma 6.6.**  $\lim_{\epsilon \rightarrow 0^+} \lambda_\epsilon = 0$ .

*Proof.* We consider the eigenvalue problem

$$\begin{cases} -\Delta \phi = \lambda b_\epsilon \epsilon^{q-2} \phi + \mu(\lambda) \phi & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.2)$$

This problem has the smallest eigenvalue  $\mu_{1,\epsilon}(\lambda)$  which satisfies

$$\begin{cases} \mu_{1,\epsilon}(\lambda) = 0, & \text{for } \lambda = 0, \lambda_\epsilon, \\ \mu_{1,\epsilon}(\lambda) > 0, & \text{for } 0 < \lambda < \lambda_\epsilon, \\ \mu_{1,\epsilon}(\lambda) < 0, & \text{for } \lambda > \lambda_\epsilon. \end{cases}$$

First we consider the case  $\int_\Omega b < 0$ . Since  $\Omega_+^b \neq \emptyset$ , the eigenvalue problem

$$\begin{cases} -\Delta \varphi = \lambda b \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique positive principal eigenvalue  $\lambda_1(b)$  with a positive eigenfunction  $\varphi_1(b)$ . It follows that  $\int_\Omega b \varphi_1(b)^2 = \frac{1}{\lambda_1(b)} \int_\Omega |\nabla \varphi_1(b)|^2 > 0$ . Then there exist  $\epsilon_1 > 0$  and  $c_1 > 0$  such that  $\int_\Omega b_\epsilon \varphi_1(b)^2 > c_1$  if  $0 < \epsilon \leq \epsilon_1$ . Let  $\lambda > 0$ . Then we deduce for such  $\epsilon$

$$\int_\Omega |\nabla \varphi_1(b)|^2 - \lambda \epsilon^{2-q} \int_\Omega b_\epsilon \varphi_1(b)^2 < \int_\Omega |\nabla \varphi_1(b)|^2 - \lambda \epsilon^{2-q} c_1.$$

Let  $\epsilon_2 = \left( \frac{\int_\Omega |\nabla \varphi_1(b)|^2}{\lambda c_1} \right)^{\frac{1}{2-q}}$ . If  $0 < \epsilon \leq \min(\epsilon_1, \epsilon_2)$  then

$$\int_\Omega |\nabla \varphi_1(b)|^2 - \lambda \epsilon^{2-q} \int_\Omega b_\epsilon \varphi_1(b)^2 < 0.$$

This implies  $\mu_{1,\epsilon}(\lambda) < 0$ , and hence,  $\lambda_\epsilon < \lambda$ , as desired.

Next we consider the case  $\int_\Omega b = 0$ . In this case, the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda b\phi + \mu(\lambda)\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

has the smallest eigenvalue  $\mu_1(\lambda)$ , which is negative for every  $\lambda > 0$ . Let  $\lambda > 0$ . Since  $\mu_1(\lambda) < 0$ , we can choose  $\phi$  such that  $\int_\Omega |\nabla\phi|^2 - \lambda \int_\Omega b\phi^2 < 0$ . Note that  $\phi$  is not a constant, so that  $\int_\Omega b\phi^2 > \frac{1}{\lambda} \int_\Omega |\nabla\phi|^2 > 0$ . Then there exist  $\epsilon_1 > 0$  and  $c_1 > 0$  such that if  $0 < \epsilon \leq \epsilon_1$  then  $\int_\Omega b_\epsilon\phi^2 > c_1$ . The rest of the proof in this case is the same as in the previous case. The proof of Lemma 6.6 is complete.  $\square$

Now Corollary 6.2 and Proposition 6.5 provide sufficient conditions under which  $\mathcal{C}_0(\epsilon)$  and  $\mathcal{C}_1(\epsilon)$  are bounded in  $\mathbb{R} \times C(\overline{\Omega})$ , and consequently coincide:

**Theorem 6.7.** *Let  $1 < q < 2 < p$  and  $\epsilon > 0$ . Assume (1.14),  $(H_0)$ ,  $(H_1)$  and  $(H_2)$ , and in addition,  $0 < \epsilon \leq \epsilon_0$ , where  $\epsilon_0$  is given by Corollary 6.2. Then the subcontinua  $\mathcal{C}_0(\epsilon)$  and  $\mathcal{C}_1(\epsilon)$  obtained in Theorem 5.1 with  $m = b_\epsilon$  are bounded in  $\mathbb{R} \times C(\overline{\Omega})$ , uniformly in  $\epsilon > 0$  small. Consequently,  $\mathcal{C}_0(\epsilon) = \mathcal{C}_1(\epsilon)$  (see Figure 2).*

*Proof.* By Corollary 6.2 and Proposition 6.5 we know that if  $u$  is a nontrivial non-negative solution of  $(P_{\lambda,\epsilon})$  then  $|\lambda| \leq \overline{\lambda}$  and  $\|u\|_{C(\overline{\Omega})} \leq C_{\overline{\lambda}}$  for some  $\overline{\lambda}$  and  $C_{\overline{\lambda}}$ . Hence the conclusion follows from Theorem 5.1(3).  $\square$

From now on we write  $\mathcal{C}_*(\epsilon) = \mathcal{C}_0(\epsilon) = \mathcal{C}_1(\epsilon)$ . As a by-product of Theorem 6.7, we determine the direction of the bifurcation  $\mathcal{C}_0(\epsilon)$  at  $(0,0)$  if  $\int_\Omega a \geq 0$ , which partially complements (5.4) and (5.5). To this end we use the following lemma:

**Lemma 6.8.** *The following two assertions hold:*

- (1) *If  $\int_\Omega a \geq 0$  then there is no positive solution of (1.11).*
- (2) *Assume  $2 < p < \frac{2N}{N-2}$  if  $N > 2$ . If  $\int_\Omega a < 0$  then there exists  $C > 0$  such that  $\|u\|_{C(\overline{\Omega})} \geq C$  for all positive solutions of (1.11).*

*Proof.*

- (1) If  $u$  is a positive solution of (1.11) then

$$\int_\Omega a = \int_\Omega \frac{-\Delta u}{u^{p-1}} = \int_\Omega |\nabla u|^2 (1-p) u^{-p} < 0.$$

- (2) Assume by contradiction that  $(u_n)$  is a sequence of positive solutions of (1.11) such that  $u_n \rightarrow 0$  in  $C(\overline{\Omega})$ . It follows that  $u_n \rightarrow 0$  in  $X$ , since  $u_n$  is a positive solution of (1.11). We set  $v_n = \frac{u_n}{\|u_n\|}$  and assume that  $v_n \rightarrow v_0$  in  $X$ ,  $v_n \rightarrow v_0$  in  $L^p(\Omega)$ , and  $v_n \rightarrow v_0$  a.e. in  $\Omega$ , for some  $v_0 \in H^1(\Omega)$ . We deduce that

$$\int_\Omega |\nabla v_0|^2 \leq \liminf_{n \rightarrow \infty} \int_\Omega |\nabla v_n|^2 \leq \limsup_{n \rightarrow \infty} \int_\Omega |\nabla v_n|^2 = \lim_{n \rightarrow \infty} \|u_n\|^{p-2} \int_\Omega a v_n^p = 0.$$

It follows that  $v_n \rightarrow v_0$  in  $X$  and  $v_0$  is a non-negative constant. Since  $\|v_n\| = 1$ , we deduce that  $v_0 \neq 0$ . However,

$$\int_\Omega a v_n^p = \|u_n\|^{2-p} \int_\Omega |\nabla v_n|^p \geq 0.$$

Passing to the limit, we get  $v_0^p \int_{\Omega} a \geq 0$ . Hence  $\int_{\Omega} a \geq 0$ , which contradicts our assumption.  $\square$

**Corollary 6.9.** *In addition to the conditions of Theorem 6.7, assume that  $\int_{\Omega} a \geq 0$ . Then  $\mathcal{C}_0(\epsilon)$  bifurcates to the region  $\lambda > 0$  at  $(0, 0)$ , that is,  $\mu(s) > 0$  for  $s > 0$  small in Theorem 5.1(1) with  $m = b_{\epsilon}$ .*

*Proof.* We argue by contradiction: in view of Lemma 6.8(1) we may assume  $\lambda_n < 0$  and a positive solution  $u_n$  of  $(P_{\lambda_n, \epsilon})$  such that  $\lambda_n \rightarrow 0^-$  and  $\|u_n\|_{C(\overline{\Omega})} \rightarrow 0$ . Theorem 5.1(1) shows that  $(\lambda_n, u_n) \in \mathcal{C}_*(\epsilon) (= \mathcal{C}_0(\epsilon))$ . Hence Theorem 6.7 ensures that  $\mathcal{C}_*(\epsilon)$  should contain  $(0, u)$  for some  $u \neq 0$ . However, this contradicts Lemma 6.8(1).  $\square$

Let us show now that under  $(H_3)$  bifurcation from zero can not occur for  $(P_{\lambda})$  at any  $\lambda \neq 0$ , i.e. it never occurs that there are  $\lambda_n \rightarrow \lambda^* \neq 0$  and nontrivial non-negative solutions  $u_n$  of  $(P_{\lambda_n})$  such that  $u_n \rightarrow 0$  in  $C(\overline{\Omega})$ . This result is deduced from Proposition 3.3:

**Proposition 6.10.** *Assume  $(H_3)$ . Then bifurcation from zero never occurs for  $(P_{\lambda})$  at any  $\lambda \neq 0$ .*

*Proof.* The assertion for  $\lambda > 0$  has been already verified in Proposition 3.3. Assume that  $\lambda_n \rightarrow \lambda^* < 0$  and  $u_n$  are nontrivial non-negative solutions of  $(P_{\lambda_n})$  with  $u_n \rightarrow 0$  in  $C(\overline{\Omega})$ . Then  $(\lambda_n, u_n)$  satisfies

$$-\Delta u_n = a u_n^{p-1} + (-\lambda_n)(-b) u_n^{q-1} \quad \text{in } \Omega.$$

Since  $-\lambda_n \rightarrow -\lambda^* > 0$  and  $\Omega_+^{-b}$  is a subdomain of  $\Omega$ , Proposition 3.3 provides us with a contradiction. The proof of Proposition 6.10 is now complete.  $\square$

We are now in position to prove Theorem 1.6.

*Proof of Theorem 1.6.* We use Whyburn's topological method [29]. Let us recall from [29] that if  $E_n \subset X$  for a complete metric space  $X$ , then

$$\begin{aligned} \liminf E_n &= \{x \in X : \lim_{n \rightarrow \infty} \text{dist}(x, E_n) = 0\}, \\ \limsup E_n &= \{x \in X : \liminf_{n \rightarrow \infty} \text{dist}(x, E_n) = 0\}. \end{aligned}$$

Note that the values  $\overline{\lambda}$  from Corollary 6.2 and  $C_{\Lambda}$  from Proposition 6.5 do not depend on the value  $\epsilon > 0$  determined in Theorem 6.7. Hence, we see that for such  $\epsilon$

$$\mathcal{C}_*(\epsilon) \subset \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : |\lambda| \leq \overline{\lambda} + 1, 0 \leq u < C_{\Lambda} + 1 \text{ on } \overline{\Omega}\}.$$

It's then clear that  $\mathcal{C}_*(\epsilon)$  is non-empty and connected, and  $(0, 0) \in \liminf \mathcal{C}_*(\epsilon)$ . Moreover, by elliptic regularity,  $\bigcup_{\epsilon} \mathcal{C}_*(\epsilon)$  is precompact. Indeed, for any  $\{(\lambda_n, u_n)\} \subset \bigcup_{\epsilon} \mathcal{C}_*(\epsilon)$  it follows that  $(\lambda_n, u_n) \in \mathcal{C}_*(\epsilon_n)$  for some  $\epsilon_n \in (0, 1]$ , so that  $u_n \in C^2(\overline{\Omega})$ , and

$$\begin{cases} -\Delta u_n = a u_n^{p-1} + \lambda_n b_{\epsilon_n} (u_n + \epsilon_n)^{q-2} u_n & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega. \end{cases}$$

Since  $|\lambda_n| \leq \overline{\lambda} + 1$  and  $\|u_n\|_{C(\overline{\Omega})} \leq C_{\Lambda} + 1$ , by elliptic regularity we find a constant  $C > 0$  such that  $\|u_n\|_{C^{1+\theta}(\overline{\Omega})} \leq C$  for all  $n$ . By the compact embedding  $C^{1+\theta}(\overline{\Omega}) \subset C(\overline{\Omega})$ , a subsequence of  $(u_n)$  converges to some  $u_0$  in  $C(\overline{\Omega})$ , as desired.

Now, by [29, Theorem 9.1], we deduce that  $\mathcal{C} := \limsup_{\epsilon \rightarrow 0^+} \mathcal{C}_*(\epsilon)$  is non-empty, closed and connected. In addition,  $\mathcal{C}$  is contained in the non-negative solutions set of  $(P_{\lambda})$ . Indeed, let  $(\lambda, u) \in \mathcal{C}$ . Then we deduce that there exists  $(\lambda_n, u_n) \in \mathcal{C}_*(\epsilon_n)$  such that

$(\lambda_n, u_n) \rightarrow (\lambda, u)$  in  $\mathbb{R} \times C(\overline{\Omega})$ . Similarly as in the above argument, elliptic regularity and Schauder estimates yield a constant  $C > 0$  such that  $\|u_n\|_{C^{2+\theta}(\overline{\Omega})} \leq C$  for all  $n$ . By a compactness argument, we deduce that a subsequence of  $(u_n)$  converges to  $u$  in  $C^2(\overline{\Omega})$ , so that  $u$  is a non-negative solution of  $(P_\lambda)$ , as desired. Furthermore, from Lemma 6.6 we infer that  $\mathcal{C}$  is a loop type subcontinuum in the sense that it bifurcates at  $(0, 0)$  and goes back to  $(0, 0)$ .

It remains to show that  $\mathcal{C}$  is nontrivial. Since  $\int_\Omega a < 0$ , assertion (5.4) with  $m = b_\epsilon$  enables us to deduce that for any  $\epsilon > 0$  small there exists  $u_\epsilon \neq 0$  such that  $(0, u_\epsilon) \in \mathcal{C}_*(\epsilon)$ . Then  $u_\epsilon$  is a positive solution of (1.11). By a standard compactness argument, there exist some sequences  $\epsilon_n$  and  $u_n := u_{\epsilon_n}$  such that  $\epsilon_n \searrow 0$  and  $u_n$  converges to a non-negative solution  $u_0$  of (1.11) in  $C(\overline{\Omega})$ . By definition, we have  $(0, u_0) \in \mathcal{C}$ . In addition, we deduce that  $u_0$  is positive on  $\overline{\Omega}$  thanks to Lemma 6.8(2). This implies that  $\mathcal{C}$  is nontrivial, i.e. it never shrinks to the  $\lambda$  axis. Finally, by Proposition 6.10 we infer that  $\mathcal{C} \setminus \{(0, 0)\}$  does not include trivial solutions  $(\lambda, 0)$  with  $\lambda \neq 0$ , and by Lemma 6.8(2) that there exists  $\delta > 0$  such that  $\mathcal{C}$  never meets any positive solution  $u$  of (1.11) satisfying  $\|u\|_{C(\overline{\Omega})} \leq \delta$ .

The proof of Theorem 1.6 is now complete.  $\square$

### 6.1. Remarks and expectations.

- (1) Theorem 1.6 remains true if we replace the condition  $\int_\Omega a < 0$  by the condition that there exists  $\epsilon_n \searrow 0$ ,  $u_n \neq 0$  such that  $\mathcal{C}_*(\epsilon_n)$  include  $(0, u_n)$ , respectively.
- (2) If instead of  $\int_\Omega a < 0$  we assume now  $\int_\Omega a \geq 0$  in Theorem 1.6 then we expect the existence of a loop type subcontinuum of nontrivial non-negative solutions of  $(P_\lambda)$  bifurcating at  $(0, 0)$  as in Figure 4. Note that, by Corollary 6.9,  $\mathcal{C}_*(\epsilon)$  bifurcates to the region  $\lambda > 0$  at  $(0, 0)$ . In addition, Lemma 6.8(1) tells us that  $\mathcal{C}_*(\epsilon)$  never meets the vertical axis  $\lambda = 0$ . The nontrivial non-negative solutions  $u_{1,\lambda}, u_{2,\lambda}$  of  $(P_\lambda)$  provided by Theorem 1.1 via a variational approach and also the non-existence result in Remark 1.5 would strongly support this suggestion. See also Remark 2.13 (2) for the case  $\int_\Omega a = 0$ . However, we couldn't exclude the possibility that  $\mathcal{C}_*(\epsilon)$  shrinks to the origin  $(0, 0)$  as  $\epsilon \rightarrow 0^+$ .
- (3) Changing  $\lambda_\epsilon$  to  $-\lambda_\epsilon$  we see that Theorems 6.7 and 1.6 hold true as well under the condition  $\Omega_-^b \neq \emptyset$  and  $\int_\Omega b > 0$ .

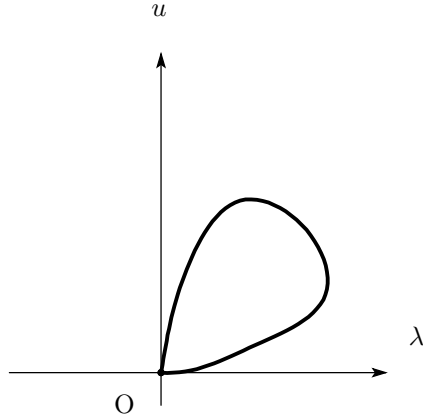


FIGURE 4. An expected loop type subcontinuum of  $(P_\lambda)$  when  $\int_\Omega a \geq 0$ ,  $\Omega_+^b \neq \emptyset$  and  $\int_\Omega b \leq 0$ .

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